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ON QUASI-CESÀRO SUMMABILITY

By A. J. WHITE (*Aberdeen*)

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1. In a recent note (9) Kuttner has defined quasi-Cesàro summability in the following way. The sequence $\{s_n\}$ is said to be *summable by quasi-Cesàro means of order* α (>0) (*summable* (C^*,α)) *to* s whenever

$$\frac{(n+1)}{\alpha+1}\sum_{m=n}^{\infty}\binom{m-n+\alpha-1}{\alpha-1}s_m\Big/\binom{m+\alpha+1}{\alpha+1}\to s, \tag{1.1}$$

as $n\to\infty$, it being assumed that the series (1.1) is convergent for each n.

Concerning the connexion between summability (C^*,α) and summability (C,α) Kuttner proved the following two theorems:

Theorem A. *If r is a positive integer, then $\{s_n\}$ is summable (C,r) to s whenever it is summable (C^*,r) to the same sum.*

Theorem B. *The proposition that the summability (C,r) to s of $\{s_n\}$ implies its summability (C^*,r) to s is false when $0<r<2$ $(r\neq 1)$ but true when $r=1$ or 2.*

Theorem B cannot be extended to the range $r>2$ since there are sequences $\{s_n\}$ summable (C,r) for which $s_n=o(n^r)$ and no more is true. For such a sequence, if $r>2$, the series (1.1) will not be convergent, and the (C^*,r) method will not be applicable.

In this paper I generalize Kuttner's definition by introducing a second parameter in the following way. I shall say that the sequence $\{s_n\}$ *is summable by quasi-Cesàro means*† *of order* (α,β) $(\alpha>0;\ \beta\geqslant 0)$ (*summable* $(C^*;\alpha,\beta)$) whenever

$$t^{\alpha,\beta}(s_n)=\frac{\beta+1}{\alpha+\beta+1}E_n^{\beta+1}\sum_{m=n}^{\infty}\frac{E_{m-n}^{\alpha-1}}{E_m^{\alpha+\beta+1}}s_m\to s, \tag{1.2}$$

as $n\to\infty$, where $E_\nu^\gamma=\binom{\nu+\gamma}{\gamma}$, and it is assumed that the series on the right of (1.2) converges for each n.

† These are the quasi-Hausdorff means generated by the moment constants

$$\binom{n+\beta}{\beta}\binom{n+\alpha+\beta}{\alpha+\beta}^{-1}.$$

The corresponding Hausdorff means have been considered by Borwein (2).

Justification for the terminology is found in Theorem 2 below where it is shown that summability $(C^*;\alpha,\beta)$ is equivalent to summability $(C^*;\alpha)$ whenever both methods are applicable.

On the other hand the sequence $\{(-)^n n^3\}$ is summable $(C^*;\alpha,\beta)$ whenever $\alpha > 3$, $\beta \geqslant 1$ whereas the $(C^*;\alpha)$ method does not apply, so that the introduction of β leads to an increase in scope.

It is easy to see that, if $\{s_n\}$ is summable (C,α), then the $(C^*;\alpha,\beta)$ method is applicable to $\{s_n\}$ whenever $\beta > \alpha-2$, and in the light of this we are able to extend Theorem B to values of r greater than 2. In fact, if $\alpha > 1$, $(C^*;\alpha,\beta)$ is applicable to $\{s_n\}$, if $s_n \to 0$ (C,α), whenever $\beta \geqslant \alpha-2$. For, when $\sum (s_n-s_{n-1})$ is summable (C,α), then $\sum (s_n-s_{n-1})n^{1-\alpha}$ is summable $(C,1)$ and hence $\sum s_n n^{-\alpha}$ is convergent [see, for example, (**4**)]. We also examine in § 5 the connexion between summability (C,α) and $(C^*;\alpha,\beta)$ when α is not an integer.

We work throughout with a 'Rieszian' definition, introduced in § 2, which we shall require elsewhere. The equivalence of this definition to the one given above occupies § 4. In § 3 we obtain a consistency theorem.

2. Notation and definitions

For a given sequence $\{u_n\}$ $(n = 0,1,\ldots)$ we write

$$S_\omega^{\alpha,\beta}(u) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(\alpha)}\omega^{\beta+1}\sum_{n>\omega}\left(1-\frac{\omega}{n}\right)^{\alpha-1}\frac{u_n}{n^{\beta+2}} \quad (\alpha \geqslant 1,\ \beta \geqslant 0) \tag{2.1}$$

subject to the convergence of the series on the right. If $S_\omega^{\alpha,\beta}(u)$ exists, we shall say that '$(R^*;\alpha,\beta)$ is applicable to $\{u_n\}$'. It is easily verified that $(R^*;\alpha,\beta)$ is applicable to $\{u_n\}$ if and only if $\sum n^{-(\beta+2)}u_n$ is convergent.

If $p \geqslant 0$, if $(R^*;\alpha,\beta)$ is applicable to $\{u_n\}$, and if

$$S_\omega^{\alpha,\beta}(u) \sim s\omega^{-p}\frac{\Gamma(\beta+p+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+p+1)\Gamma(\beta+1)},$$

as $\omega \to \infty$, then we shall write

$$u_n \sim sn^{-p}(R^*;\alpha,\beta),$$

as $n \to \infty$, and the related notation such as $u_n = o(n^{-p})$ $(R^*;\alpha,\beta)$ is interpreted in the obvious way. In particular, if $s_n \sim s$ $(R^*;\alpha,\beta)$, where $\alpha \geqslant 1$, $\beta \geqslant 0$, and $s_n = \sum_{\nu=0}^{n} a_\nu$ we shall say that $\sum_{\nu=0}^{\infty} a_\nu$ is 'summable $(R^*;\alpha,\beta)$ to s'. This definition may be extended to the case $0 < \alpha < 1$

as follows. If $0 < \alpha < 1$ and $\beta \geqslant 0$, we say that $\sum_{\nu=0}^{\infty} a_\nu$ 'is summable $(R^*; \alpha, \beta)$ to s' whenever

(i) $na_n = o(1)\ (R^*; \alpha+1, \beta)$,

(ii) $\sum_{n=0}^{\infty} a_n$ is summable $(R^*; \alpha+1, \beta)$ to s.

The consistency of these two definitions is shown by the following theorem:

THEOREM 1. *If $\alpha \geqslant 1$ and $\beta \geqslant 0$, then in order that $\sum a_\nu$ be summable $(R^*; \alpha, \beta)$ to s it is necessary and sufficient that $na_n = o(1)\ (R^*; \alpha+1, \beta)$ and $\sum a_\nu$ be summable $(R^*; \alpha+1, \beta)$ to s.*

A proof of this is given in Lemma 8, below.

If $(R^*; \alpha, \beta)$ $(\alpha \geqslant 1, \beta \geqslant 0)$ is applicable to $\{s_\nu\}$ we write

$$S_\beta(x) = \sum_{\nu=n+1}^{\infty} \frac{s_\nu}{\nu^{\beta+2}} \quad (n < x \leqslant n+1), \tag{2.2}$$

$$T_{\alpha,\beta}(x) = x^{-\alpha} S_\beta(x) \quad (x > 0). \tag{2.3}$$

It is then easily verified that, for $\alpha > 1$,

$$S_\omega^{\alpha,\beta}(s) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha-1)\Gamma(\beta+1)} \omega^{\beta+1} \int_\omega^\infty \left(1-\frac{\omega}{x}\right)^{\alpha-2} S_\beta(x) \frac{dx}{x^2}, \tag{2.4}$$

and that, since $S_\beta(x) = o(1)$ as $x \to \infty$,

$$I(\gamma, x) = \frac{1}{\Gamma(\gamma-1)} \int_x^\infty (t-x)^{\gamma-2} T_{\alpha,\beta}(t)\, dt \tag{2.5}$$

exists for all x if $\alpha > \gamma-1 > 0$. A standard argument† shows that for $\alpha > \gamma-1 > \delta-1 > 0$,

$$I(\gamma, x) = \frac{1}{\Gamma(\gamma-\delta)} \int_x^\infty (t-x)^{\gamma-\delta-1} I(\delta, t)\, dt, \tag{2.6}$$

whenever the left-hand side exists.

We require also the means $\bar{t}^{\alpha,\beta}(u_n)$ defined for the given sequence $\{u_n\}$ by

$$\bar{t}^{\alpha,\beta}(u_n) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} n^{\beta+1} \sum_{m=n}^{\infty} E_{m-n}^{\alpha-1} \frac{u_m}{m^{\alpha+\beta+1}} \quad (\alpha > 0, \beta \geqslant 0),$$

it being assumed that the series on the right is convergent. Summability

† See, for example, (3) 142. We define, formally,

$$I(1, x) = S_{\alpha,\beta}(x).$$

$(\bar{C};\alpha,\beta)$, and related ideas, are defined in terms of these means in a similar way to those involving the $(R^*;\alpha,\beta)$ process.

Finally, throughout this paper c, c_0, c_1,... denote constants not necessarily the same at each occurrence, and $\sum$ denotes $\sum\limits_0^\infty$.

3. In this section we prove the following theorem:

THEOREM 2. *If $\sum a_n$ is summable $(R^*;\alpha,\beta)$ $(\beta \geqslant 0)$ to s, then it is summable $(R^*;a,b)$ $(b \geqslant 0)$ to s whenever $a \geqslant \alpha > 0$ and $(R^*;a,b)$ is applicable to $\sum a_n$.*

Since, by Theorem 3 below, $(R^*;\alpha,\beta)$ is equivalent to $(C^*;\alpha,\beta)$, there is an analogue of Theorem 2 for summability $(C^*;\alpha,\beta)$. It is in this analogue that we find justification for regarding $(C^*;\alpha,\beta)$ as a natural extension of (C^*,α). We require first a lemma [see (**8**) 683]:

LEMMA 1. *If $0 < x \leqslant y \leqslant 2x$ and if $s_n \to 0$ $(R^*;\alpha,\beta)$ $(\beta \geqslant 0)$, where α (>1) is not an integer, then*

$$\int\limits_x^y I([\alpha],t)\,dt = o\{y^{-\beta-2}(y-x)^{[\alpha]-\alpha+1}\},$$

as $x \to \infty$, where $[\alpha]$ is the integral part of α.

We first observe that, from (2.4) and (2.5), $s_n \to 0$ $(R^*;\alpha,\beta)$ is equivalent to

$$I(\alpha,x) = o(x^{-\beta-2}) \quad (x \to \infty). \tag{3.1}$$

Next, by (2.6),

$$I([\alpha]+1,x) = c\int\limits_x^\infty (t-x)^{[\alpha]-\alpha}I(\alpha,t)\,dt,$$

so that

$$\begin{aligned} c\int\limits_x^y I([\alpha],t)\,dt &= I([\alpha]+1,x)-I([\alpha]+1,y) \\ &= c_1\int\limits_x^y I(\alpha,t)(t-x)^{[\alpha]-\alpha}\,dt- \\ &\qquad -c_1\int\limits_y^\infty I(\alpha,t)\{(t-y)^{[\alpha]-\alpha}-(t-x)^{[\alpha]-\alpha}\}\,dt. \end{aligned}$$

By (3.1) the first term is

$$o\left\{\int\limits_x^y (t-x)^{[\alpha]-\alpha}t^{-\beta-2}\,dt\right\} = o\{y^{-\beta-2}(y-x)^{[\alpha]-\alpha+1}\},$$

while the second is

$$o\left\{\int\limits_y^\infty t^{-\beta+[\alpha]-\alpha}\left[\left(1-\frac{y}{t}\right)^{[\alpha]-\alpha}-\left(1-\frac{x}{t}\right)^{[\alpha]-\alpha}\right]\frac{dt}{t^2}\right\}$$

$$= o\left\{y^{-\beta+[\alpha]-\alpha}\int\limits_y^\infty\left[\left(1-\frac{y}{t}\right)^{[\alpha]-\alpha}-\left(1-\frac{x}{t}\right)^{[\alpha]-\alpha}\right]dt\right\}\quad (x\leqslant y)$$

$$= o\{y^{-\beta-2}(y-x)^{[\alpha]-\alpha+1}\}.$$

Proof of Theorem 2. *Case* 1: $\alpha \geqslant 1$. We may suppose, without loss of generality that $s = 0$.† Since $(R^*; a, b)$ is applicable to $\sum a_n$, $S_\omega^{a,b}(s)$ exists and we have

$$S_\omega^{a,b}(s) = c\omega^{b+1}\sum_{n>\omega}\left(1-\frac{\omega}{n}\right)^{a-1}\frac{s_n}{n^{b+2}}$$

$$= c\omega^{b+1}\sum_{n>\omega}\left(1-\frac{\omega}{n}\right)^{a-1} n^{\beta-b}\frac{s_n}{n^{\beta+2}}$$

$$= c\omega^{b+1}\int\limits_\omega^\infty S_\beta(x)\frac{d}{dx}\left\{\left(1-\frac{\omega}{x}\right)^{a-1}x^{\beta-b}\right\}dx$$

$$= c\omega^{b+1}\int\limits_\omega^\infty S_\beta(x)(x-\omega)^{a-2}x^{\beta-b-a+1}\,dx+$$

$$+c_1\,\omega^{b+1}\int\limits_\omega^\infty S_\beta(x)(x-\omega)^{a-1}x^{\beta-b-a}\,dx$$

$$= I_1+I_2,\quad \text{say.}$$

We prove that $I_1 = o(1)$ as $\omega\to\infty$; a similar argument shows that $I_2 = o(1)$, and this is enough to prove Case 1.

Integrating by parts we have‡

$$I_1 = \omega^{b+1}\sum_{\nu=1}^{[\alpha]-1}c_\nu\left[I(\nu+1,x)\left(\frac{d}{dx}\right)^{\nu-1}\{(x-\omega)^{a-2}x^{\beta-b+\alpha-a+1}\}\right]_\omega^\infty+$$

$$+c\omega^{b+1}\int\limits_\omega^\infty I([\alpha],x)\left(\frac{d}{dx}\right)^{[\alpha]-1}\{(x-\omega)^{a-2}x^{\beta-b+\alpha-a+1}\}\,dx.$$

Since $S_\beta(x) = o(1)$ $(S_\beta(x) = o(x^{b-\beta})$ if $b<\beta)$, it follows that

$$I(\nu,x) = o(x^{\nu-\alpha-1})\quad (I(\nu,x) = o(x^{\nu-\alpha+1+b-\beta}) \text{ if } b<\beta)$$

for $\nu = 1, 2,\ldots, [\alpha]+1$, and hence that the summation term vanishes,

† We also exclude the case $\alpha = a = 1$, the result then being obtainable by a single summation by parts.

‡ If $[\alpha] = 0$, the summation term vanishes.

unless $\alpha = a$ is an integer, when it is $c\omega^{\beta+2}I(\alpha,\omega) = o(1)$ by (3.1), i.e.†

$$I_1 = \omega^{b+1}\sum_{p=0}^{[\alpha]-1} c_p \int_{\omega}^{\infty} I([\alpha],x)(x-\omega)^{a-2-p}x^{\beta-b+\alpha-a-[\alpha]+p+2}\,dx. \tag{3.2}$$

If α is an integer, then, since our hypothesis is equivalent to (3.1), it follows immediately from (3.2) that $I_1 = o(1)$. If α is not an integer, we write (3.2) as

$$I_1 = \sum_{p=0}^{[\alpha]-1} c_p\{J_{p,1}+J_{p,2}\},$$

where

$$J_{p,1} = \lim_{\eta\to+\omega} \omega^{b+1}\int_{\eta}^{2\omega} I([\alpha],x)(x-\omega)^{a-2-p}x^{\beta-b+\alpha-a-[\alpha]+p+2}\,dx$$

$$= \lim_{\eta\to+\omega}\Biggl\{\omega^{b+1}\Biggl[(x-\omega)^{a-2-p}x^{\beta-b+\alpha-a-[\alpha]+p+2}\int_{\omega}^{x} I([\alpha],t)\,dt\Biggr]_{x=\eta}^{2\omega}+$$

$$+\omega^{b+1}\int_{\eta}^{2\omega}\frac{d}{dx}\{(x-\omega)^{a-2-p}x^{\beta-b+\alpha-a-[\alpha]+p+2}\}\,dx\int_{\omega}^{x} I([\alpha],t)\,dt\Biggr\}$$

$$= o(1),$$

by Lemma 1, and

$$J_{p,2} = \omega^{b+1}\int_{2\omega}^{\infty} I([\alpha],x)(x-\omega)^{a-2-p}x^{\beta-b+\alpha-a-[\alpha]+p+2}\,dx$$

$$= \omega^{b+1}[I([\alpha]+1,x)(x-\omega)^{a-2-p}x^{\beta-b+\alpha-a-[\alpha]+p+2}]_{x=2\omega}^{\infty}+$$

$$+c\omega^{b+1}\int_{2\omega}^{\infty} I([\alpha]+1,x)\frac{d}{dx}\{(x-\omega)^{a-2-p}x^{\beta-b+\alpha-a-[\alpha]+p+2}\}\,dx$$

$$= o(1)$$

since $I([\alpha]+1,x) = o(x^{[\alpha]-\alpha-\beta-1})$.

Case 2: $0 < \alpha < 1$. In this case our hypothesis is that $\sum a_n$ is summable $(R^*;\alpha+1,\beta)$ to s, and that $na_n \to 0$ $(R^*;\alpha+1,\beta)$. Since $\alpha+1 > 1$, it follows from Case 1 that $\sum a_n$ is summable $(R^*;a+1,b)$ to s and again from Case 1, with s_n replaced by na_n, that

$$na_n \to 0 \quad (R^*;a+1,b),$$

i.e. that $\sum a_n$ is summable $(R^*;a,b)$ to s.

† Certain of the c_p in the expression that follows may be zero if $\beta-b+\alpha-a+1$ is a positive integer or zero or if $\alpha = a =$ an integer.

4. We now show that summability $(C^*; \alpha, \beta)$ and $(R^*; \alpha, \beta)$ are equivalent by the theorem:

THEOREM 3. *If $\alpha > 0$, $\beta \geqslant 0$, then $\sum a_n$ is summable $(C^*; \alpha, \beta)$ to s if and only if it is summable $(R^*; \alpha, \beta)$ to s.*

We require a number of lemmas:

LEMMA 2. *If $\alpha > 0$ and $\beta \geqslant 0$ and if $(C^*; \alpha, \beta)$ is applicable to $\sum a_n$, then*

$$\frac{(\beta+1)}{\alpha+\beta+1} s_n = E_n^{\alpha+\beta+1} \sum_{m=n}^{\infty} E_{m-n}^{-\alpha-1} t^{\alpha,\beta}(s_m)/E_m^{\beta+1}.$$

This follows from a theorem due to Andersen [(**1**) Theorem I].

We deduce from Lemma 2 the following 'limitation condition' for summability $(C^*; \alpha, \beta)$:

COROLLARY. *If $\sum a_n$ is summable $(C^*; \alpha, \beta)$ to s, where $\alpha > 0$, $\beta \geqslant 0$, then*

$$s_n - s = o(n^\alpha),$$

as $n \to \infty$.

LEMMA 3.† *Let k be the integral part of α (> 1). Let $f(x)$ and its first $k+1$ derivatives be continuous and such that*

$$f(0) = f'(0) = \dots = f^{(k)}(0) = 0$$

and

$$f(x) = E_{x-1}^{\alpha-1} \quad (x \geqslant 1).$$

Then, if

$$U(x) \equiv \int_0^x (x-t)^{k-\alpha} f^{(k+1)}(t)\, dt,$$

we have

$$U(x) = \begin{cases} O(x^{-2}) & \text{as } x \to \infty, \\ O(x^{k+1-\alpha}) & \text{as } x \to +0 \end{cases} \tag{4.1}$$

and

$$\bar{t}^{\alpha,\beta}(s_m) = cm^{\beta+1} \int_{m-1}^{\infty} S_t^{\alpha,\beta}(s) U(t-m+1) \frac{dt}{t^{\beta+1}} \tag{4.2}$$

whenever $(R^*; \alpha, \beta)$ $(\beta \geqslant 0)$ is applicable to $\sum a_n$.

Gergen has obtained (4.1). For (4.2) we observe first that, since $(R^*; \alpha, \beta)$ is applicable to $\sum a_n$, $t^{-\beta-1} S_t^{\alpha,\beta}(s) = o(1)$ and hence that the integral on the right of (4.2) is absolutely convergent. Denoting it by

† Lemmas 3 and 4 are based on working due to Gergen [(**6**) Theorems I and II]. The present case is slightly more complicated since infinite integrals and series are involved.

I_m we have

$$I_m = \int_0^\infty U(t) S^{\alpha,\beta}_{t+m-1}(s) \frac{dt}{(t+m-1)^{\beta+1}}$$

$$= c \sum_{p=0}^\infty \int_p^{p+1} U(t) \sum_{\nu=p+m}^\infty \left(1-\frac{t+m-1}{\nu}\right)^{\alpha-1} \frac{s_\nu}{\nu^{\beta+2}}\, dt$$

$$= c \sum_{p=0}^\infty \sum_{\nu=p+m}^\infty \frac{s_\nu}{\nu^{\beta+2}} \int_p^{p+1} U(t)\left(1-\frac{t+m-1}{\nu}\right)^{\alpha-1} dt,$$

where the interchange of summation and integration is justified by bounded convergence since $\{1-(t+m-1)/\nu\}^{\alpha-1}$ is a steadily increasing function of ν, so that

$$\left|\sum_{\nu=p+m}^\infty \left(1-\frac{t+m-1}{\nu}\right)^{\alpha-1} \frac{s_\nu}{\nu^{\beta+2}}\right| \leqslant 3 \max_{r \geqslant p+m} \left|\sum_{\nu=r}^\infty \frac{s_\nu}{\nu^{\beta+2}}\right|$$

for $p \leqslant t \leqslant p+1$. We thus have

$$I_m = c \sum_{\nu=m}^\infty \frac{s_\nu}{\nu^{\beta+2}} \sum_{p=0}^{\nu-m} \int_p^{p+1} U(t)\left(1-\frac{t+m-1}{\nu}\right)^{\alpha-1} dt$$

$$= c \sum_{\nu=m}^\infty \frac{s_\nu}{\nu^{\alpha+\beta+1}} \int_0^{\nu-m+1} (\nu-m+1-t)^{\alpha-1} U(t)\, dt$$

$$= c \sum_{\nu=m}^\infty \frac{s_\nu}{\nu^{\alpha+\beta+1}} f(\nu-m+1)$$

$$= c \sum_{\nu=m}^\infty E^{\alpha-1}_{\nu-m} \frac{s_\nu}{\nu^{\alpha+\beta+1}}$$

$$= cm^{-\beta-1} \bar{t}^{\alpha,\beta}(s_m), \tag{4.3}$$

where the inversion is justified since $\{1-(t+m-1)/\nu\}^{\alpha-1}$ is a steadily increasing function of ν and $U(t) = O(t^{-2})$, so that

$$\sum_{p=0}^{N-m} \sum_{\nu=N+1}^\infty \frac{s_\nu}{\nu^{\beta+2}} \int_p^{p+1} U(t)\left(1-\frac{t+m-1}{\nu}\right)^{\alpha-1} dt = \sum_{p=0}^{N-m} O\left\{\frac{1}{p^2} \max_{\nu \geqslant N+1} \left|\sum_{r=\nu}^\infty \frac{s_r}{r^{\beta+2}}\right|\right\}$$

$$= o(1),$$

as $N \to \infty$. The lemma follows from (4.3).

LEMMA 4. *Suppose that* $\alpha > 1$, $\beta \geqslant 0$, *and let*

$$\phi(x) = \sum_{p<x} E_p^{-\alpha-1}$$

and

$$V(x) = \int_0^x (x-t)^{\alpha-2}\phi(t)\,dt,$$

then

$$V(x) = \begin{cases} O(x^{-2}) & \textit{as } x \to \infty, \\ O(x^{\alpha-1}) & \textit{as } x \to +0 \end{cases} \tag{4.4}$$

and

$$S_\omega^{\alpha,\beta}(s) = c\omega^{\beta+1} \sum_{\nu \geqslant \omega} \bar{t}^{\alpha,\beta}(s_\nu) V(\nu-\omega)\nu^{-\beta-1}, \tag{4.5}$$

whenever $(\bar{C};\alpha,\beta)$ *is applicable to* $\sum a_n$.

Again (4.4) has been obtained by Gergen. To prove (4.5) we observe first that, since $(\bar{C};\alpha,\beta)$ is applicable to $\sum a_n$, $\nu^{-\beta-1}\bar{t}^{\alpha,\beta}(s_\nu)$ is bounded and it follows from (4.4) that the series on the right of (4.5) is absolutely convergent. Denoting its sum by $T(\omega)$ we have†

$$T(\omega) = c \sum_{\nu \geqslant \omega} V(\nu-\omega) \sum_{n=\nu}^{\infty} E_{n-\nu}^{\alpha-1} \frac{s_n}{n^{\alpha+\beta+1}}$$

$$= c \sum_{n=[\omega]'+1}^{\infty} \frac{s_n}{n^{\alpha+\beta+1}} \sum_{\nu=[\omega]'+1}^{n} E_{n-\nu}^{\alpha-1} V(\nu-\omega), \tag{4.6}$$

where the inversion is justified since $n^{1-\alpha}E_{n-\nu}^{\alpha-1}$ is a decreasing function of n, so that

$$\left| \sum_{\nu=[\omega]'+1}^{N} V(\nu-\omega) \sum_{n=N+1}^{\infty} E_{n-\nu}^{\alpha-1} \frac{s_n}{n^{\alpha+\beta+1}} \right|$$

$$\leqslant 3 \max_{n \geqslant N+1} \left| \sum_{r=n}^{\infty} \frac{s_r}{r^{\beta+2}} \right| \sum_{\nu=[\omega]'+1}^{N} |V(\nu-\omega)| = o(1) \sum_{\nu=[\omega]'+1}^{N} O(\nu-\omega)^{-2}$$

as $N \to \infty$. The lemma now follows from (4.6) since

$$V(\nu-\omega) = \sum_{p<\nu-\omega} (\nu-\omega-p)^{\alpha-1} E_p^{-r-1},$$

so that

$$\sum_{\nu=[\omega]'+1}^{n} E_{n-\nu}^{\alpha-1} V(\nu-\omega)$$

$$= \sum_{\nu=[\omega]'+1}^{n} E_{n-\nu}^{\alpha-1} \sum_{p=0}^{\nu-[\omega]'-1} (\nu-\omega-p)^{\alpha-1} E_p^{-\alpha-1}$$

$$= \sum_{p=0}^{n-[\omega]'-1} (p+[\omega]'-\omega+1)^{\alpha-1} \sum_{\nu=p+[\omega]'+1}^{n} E_{n-\nu}^{\alpha-1} E_{\nu-[\omega]'-1-p}^{-\alpha-1}$$

$$= (n-\omega)^{\alpha-1} \tag{4.7}$$

since the inner sum of (4.7) is zero unless $n = p+[\omega]'+1$, when it is 1.

† Here $[\omega]'$ denotes the greatest integer less than ω.

Lemma 5. *If $\alpha \geqslant 1$, $\beta \geqslant 0$ and $p \geqslant 0$, then*

$$s_n = o(n^{-p}) \quad (R^*; \alpha, \beta)$$

if and only if
$$s_n = o(n^{-p}) \quad (\bar{C}; \alpha, \beta).$$

This is an immediate consequence of Lemmas 3 and 4, unless $\alpha = 1$, when it is trivial.

Lemma 6. *If $\alpha \geqslant 1$ and $\beta \geqslant 0$, if r is a positive integer, and if $s_n \to 0$ $(\bar{C}; \alpha, \beta)$, then $n^{-r}s_n = o(n^{-r})$ $(\bar{C}; \alpha, \beta)$.*

The identity

$$\left(1-\frac{\omega}{n}\right)^{\alpha-1}\frac{\omega^r}{n^r} = \sum_{\mu=0}^{r}\binom{r}{\mu}\left(1-\frac{\omega}{n}\right)^{\alpha+\mu-1}$$

is easily verified, and it follows that

$$\omega^r S_\omega^{\alpha,\beta}(t) = \sum_{\mu=0}^{r} c_\mu\, S_\omega^{\alpha+\mu,\beta}(s) \quad (t = \{n^{-r}s_n\}).$$

The result now follows from Lemma 5 and the consistency of (R^*) means.

Lemma 7. *If $0 < \alpha < 1$ and $\beta \geqslant 0$ and if $\sum a_n$ is summable $(C^*; \alpha, \beta)$ to s, then it is summable $(C^*; 1, \beta)$ to s.*

We may suppose that $s = 0$. We then have

$$\begin{aligned} t^{1,\beta}(s_n) &= cE_n^{\beta+1}\sum_{m=n}^{\infty} s_m/E_m^{\beta+2} \\ &= cE_n^{\beta+1}\sum_{m=n}^{\infty} E_m^{\alpha+\beta+1}/E_m^{\beta+2}\sum_{\nu=m}^{\infty} E_{\nu-m}^{-\alpha-1}t^{\alpha,\beta}(s_\nu)/E_\nu^{\beta+1}, \text{ by Lemma 3,} \\ &= cE_n^{\beta+1}\sum_{\nu=n}^{\infty} t^{\alpha,\beta}(s_\nu)/E_\nu^{\beta+1}\sum_{m=n}^{\nu} E_m^{\alpha+\beta+1}E_{\nu-m}^{-\alpha-1}/E_m^{\beta+2}, \end{aligned} \tag{4.8}$$

where the inversion is justified since $t^{\alpha,\beta}(s_\nu) = O(1)$, so that

$$\sum_{\nu=N+1}^{\infty} E_{\nu-m}^{-\alpha-1}t^{\alpha,\beta}(s_\nu)/E_\nu^{\beta+1} = O\{N^{-\beta-1}(N+1-m)^{-\alpha}\}$$

and hence, for each fixed n,

$$\begin{aligned} \sum_{m=n}^{N} E_m^{\alpha+\beta+1}/E_m^{\beta+2} & \sum_{\nu=N+1}^{\infty} E_{\nu-m}^{-\alpha-1}t^{\alpha,\beta}(s_\nu)/E_\nu^{\beta+1} \\ &= \sum_{m=n}^{N} O\{m^{-\alpha+1}N^{-\beta-1}(N+1-m)^{-\alpha}\} \\ &= o(1) \end{aligned}$$

as $N \to \infty$. Putting $s_n \equiv 1$ in (4.8) we have

$$\sum_{\nu=n}^{\infty} 1/E_\nu^{\beta+1}\sum_{m=n} E_m^{\alpha+\beta+1}E_{\nu-m}^{-\alpha-1}/E_m^{\beta+2} = c/E_n^{\beta+1}. \tag{4.9}$$

On the other hand, the inner sum of (4.9) is, by partial summation,

$$\sum_{m=0}^{n-\nu} E_m^{-\alpha}\Delta(E_{m+\nu}^{\alpha+\beta+1}/E_{m+\nu}^{\beta+2})+E_{n-\nu}^{-\alpha}E_{n+1}^{\alpha+\beta+1}/E_{n+1}^{\beta+2},$$

which is positive since $E_{m+\nu}^{\alpha+\beta+1}/E_{m+\nu}^{\beta+2}$ is a decreasing function of m for $0<\alpha<1$. The result now follows easily from (4.8) and (4.9).

Proof of Theorem 3. We may suppose that $s = 0$.

Case 1: $\alpha \geqslant 1$. Suppose that $\sum a_n$ is summable $(R^*; \alpha, \beta)$ to zero. By Lemma 6 (with $p = 0$) $\bar{t}^{\alpha,\beta}(s_n) = o(1)$ as $n \to \infty$.

Now, if k is a positive integer,

$$s_m\left\{\frac{m^{\alpha+\beta+1}}{E_m^{\alpha+\beta+1}}-\frac{1}{\Gamma(\alpha+\beta+1)}\right\} = \left\{\sum_{\mu=1}^{k} c_\mu m^{-\mu}+O(m^{-k-1})\right\}s_m.$$

Taking the $(\bar{C}; \alpha, \beta)$ mean of both sides, we have

$$\frac{n^{\beta+1}}{E_n^{\beta+1}}t^{\alpha,\beta}(s_n)-c\bar{t}^{\alpha,\beta}(s_n)$$

$$=\sum_{\mu=1}^{k} c_\mu \bar{t}^{\alpha,\beta}\left(\frac{s_m}{m^\mu}\right)+n^{\beta+1}\sum_{m=n}^{\infty} E_{m-n}^{\alpha-1}\frac{s_m}{m^{\alpha+\beta+1}}O\left(\frac{1}{m^{k+1}}\right).$$

Since $s_m = o(m^{\beta+2})$, we can choose k so large that the last term on the right is $o(1)$. Since also, by Lemma 6,

$$\bar{t}^{\alpha,\beta}\left(\frac{s_m}{m^\mu}\right) = o(m^{-\mu}) \quad (\mu = 0, 1, 2,...),$$

it follows that $t^{\alpha,\beta}(s_n) = o(1)$ as $n \to \infty$; i.e. $\sum a_n$ is summable $(C^*; \alpha, \beta)$ to zero.

Conversely, suppose that $\sum a_n$ is summable $(C^*; \alpha, \beta)$ to zero. Since

$$\bar{t}^{\alpha,\beta}\left(\frac{m^{\alpha+\beta+1}s_m}{E_m^{\alpha+\beta+1}}\right) = c\frac{m^{\beta+1}}{E_m^{\beta+1}}t^{\alpha,\beta}(s_m),$$

it follows that $\left(\frac{m^{\alpha+\beta+1}s_m}{E_m^{\alpha+\beta+1}}\right)$ is summable $(\bar{C}; \alpha, \beta)$ to zero, and hence, by Lemma 6, that

$$\bar{t}^{\alpha,\beta}\left(\frac{m^{\alpha+\beta+1-\mu}s_m}{E_m^{\alpha+\beta+1}}\right) = o(m^{-\mu}), \tag{4.10}$$

as $m \to \infty$ for $\mu = 0, 1,...$. Further, when k is a positive integer,

$$\left\{\frac{E_m^{\alpha+\beta+1}}{m^{\alpha+\beta+1}}-\Gamma(\alpha+\beta+1)\right\}s_m = \left\{\sum_{\mu=1}^{k} c_\mu m^{-\mu}+O(m^{-k-1})\right\}s_m$$

so that, taking the $(C^*; \alpha, \beta)$ mean of both sides, for sufficiently large k, we have

$$\frac{E_m^{\beta+1}}{m^{\beta+1}}\bar{t}^{\alpha,\beta}(s_m)+ct^{\alpha,\beta}(s_m) = \frac{E_m^{\beta+1}}{m^{\beta+1}}\sum_{\mu=1}^{k} c_\mu \bar{t}^{\alpha,\beta}\left(\frac{m^{\alpha+\beta+1-\mu}s_m}{E_m^{\alpha+\beta+1}}\right)+o(1). \tag{4.11}$$

The result now follows from (4.10) and (4.11).

Before proving the case $0 < \alpha < 1$ we require a further lemma.

LEMMA 8. *If $\alpha > 0$ and $\beta \geqslant 0$, then in order that $\sum a_n$ be summable $(C^*; \alpha, \beta)$ to s it is necessary and sufficient that*

$$na_n \to 0 \quad (C^*; \alpha+1, \beta)$$

and $\sum a_n$ be summable $(C^*; \alpha+1, \beta)$ to s.

We remark that, in view of what we have already proved of Theorem 3, this lemma implies Theorem 1.

We suppose that $s = 0$. Then

$$t^{\alpha+1,\beta}(na_n) = cE_n^{\beta+1} \sum_{m=n}^{\infty} E_{m-n}^{\alpha} ma_m / E_m^{\alpha+\beta+2}$$

$$= cE_n^{\beta+1} \sum_{m=n}^{\infty} E_{m-n}^{\alpha} a_m / E_m^{\alpha+\beta+1} + c_1 E_n^{\beta+1} \sum_{m=n}^{\infty} E_{m-n}^{\alpha} a_m / E_m^{\alpha+\beta+2}. \tag{4.12}$$

By partial summation, the first term on the right is

$$cE_n^{\beta+1}\left\{-\sum_{m=n}^{\infty} s_m \Delta\left(\frac{E_{m-n}^{\alpha}}{E_m^{\alpha+\beta+1}}\right) - \frac{s_{n-1}}{E_n^{\alpha+\beta+1}}\right\}$$

$$= cE_n^{\beta+1}\left\{-\sum_{m=n}^{\infty} E_{m+1-n}^{\alpha-1} s_m / E_m^{\alpha+\beta+1} + \right.$$

$$\left. + c_1 \sum_{m=n}^{\infty} E_{m-n}^{\alpha} s_m / E_m^{\alpha+\beta+2} - s_{n-1} / E_n^{\alpha+\beta+1}\right\}$$

$$= c_1 t^{\alpha,\beta}(s_{n-1}) + c_2 t^{\alpha+1,\beta}(s_n) + s_{n-1}\left(\frac{1}{E_{n-1}^{\alpha+\beta+1}} - \frac{1}{E_n^{\alpha+\beta+1}}\right) E_n^{\beta+1}$$

$$= c_1 t^{\alpha,\beta}(s_{n-1}) + c_2 t^{\alpha+1,\beta}(s_n) + cs_{n-1} E_n^{\beta+1} / E_{n-1}^{\alpha+\beta+2}. \tag{4.13}$$

Performing a similar calculation with the second term on the right of (4.12), we have

$$t^{\alpha+1,\beta}(na_n) + O(n^{-\alpha-1} s_{n-1}) = c_1 t^{\alpha,\beta}(s_{n-1}) + c_2 t^{\alpha+1,\beta}(s_n) +$$

$$+ O\left(\frac{1}{n}\right)\{t^{\alpha,\beta+1}(s_{n-1}) + t^{\alpha+1,\beta+1}(s_n)\}. \tag{4.14}$$

Suppose first that $t^{\alpha,\beta}(s_n) = o(1)$. It follows (from Theorem 2 and Case 1 of Theorem 3 if $\alpha \geqslant 1$, and Lemma 7, Case 1 of Theorem 3, and Theorem 2 if $0 < \alpha < 1$) that $t^{\alpha+1,\beta}(s_n) = o(1)$, so that by Theorem 2 and Case 1, we have also $t^{\alpha+1,\beta+1}(s_n) = o(1)$. Further, for suitably

chosen c_1 and c_2,

$$\frac{c_1}{E_n^{\beta+1}} t^{\alpha+1,\beta}(s_n) + \frac{c_2}{E_n^{\beta+2}} t^{\alpha,\beta+1}(s_n) = c \sum_{m=n}^{\infty} (E_{m-n}^{\alpha} - E_{m-n}^{\alpha-1}) \frac{s_m}{E_m^{\alpha+\beta+2}}$$

$$= c \sum_{m=n+1}^{\infty} E_{m-n-1}^{\alpha} \frac{s_m}{E_m^{\alpha+\beta+2}} = \frac{c}{E_n^{\beta+1}} t^{\alpha+1,\beta}(s_{n+1}).$$

Consequently $t^{\alpha,\beta+1}(s_n) = o(n)$ and it follows from (4.14) that

$$t^{\alpha+1,\beta}(na_n) = o(1)$$

since, by the corollary to Lemma 2, $s_n = o(n^{\alpha})$.

Conversely, suppose that $t^{\alpha+1,\beta}(na_n) = o(1)$ and $t^{\alpha+1,\beta}(s_n) = o(1)$. It follows from Lemma 5, Lemma 6, and Case 1 of Theorem 3 that

$$t^{\alpha+1,\beta}(a_n) = o(n^{-1}).$$

Consequently, from (4.13),

$$t^{\alpha,\beta}(s_{n-1}) = ct^{\alpha+1,\beta}(s_n) + c_1 t^{\alpha+1,\beta}(a_n) + O(n^{-\alpha-1}s_n)$$

$$= o(1)$$

since, by the corollary to Lemma 2, $s_n = o(n^{\alpha+1})$.

We can now complete the proof of Theorem 3.

Case 2: $0 < \alpha < 1$. Suppose that $\sum a_n$ is summable $(R^*; \alpha, \beta)$ to zero, i.e. that $na_n \to 0$ $(R^*; \alpha+1, \beta)$ and $\sum a_n$ is summable $(R^*; \alpha+1, \beta)$. By Case 1 this implies that $na_n \to 0$ $(C^*; \alpha+1, \beta)$ and $\sum a_n$ is summable $(C^*; \alpha+1, \beta)$ to zero; i.e., by Lemma 8, $\sum a_n$ is summable $(C^*; \alpha, \beta)$ to zero.

The converse argument is similar.

5. We now examine the connexion between summability $(C^*; \alpha, \beta)$ and summability (C, α). We have the theorems:

THEOREM 4. *If* $\sum a_n$ *is summable* (C, α) *to* s, *then it is summable* $(C^*; \gamma, \delta)$ *to* s, *where* $\gamma \geqslant \alpha > 0$ *if* α *is an integer, and* $\gamma > \alpha > 0$ *otherwise, provided that* $(C^*; \gamma, \delta)$ *applies to* $\sum a_n$.

THEOREM 5. *If* $\sum a_n$ *is summable* $(C^*; \alpha, \delta)$ *to* s, *then it is summable* (C, γ) *to* s *where* $\gamma \geqslant \alpha > 0$ *if* α *is an integer and* $\gamma > \alpha > 0$ *otherwise.*

We consider only the case $\alpha > 1$ of these theorems. The results when $0 < \alpha \leqslant 1$ are deduced from this case in the usual way.

We require two further lemmas:

Lemma 9.† *If* $\alpha \geqslant 1$ *and* $\sum_{n=1}^{\infty} s_n n^{-\beta-2}$ *is convergent, then* $s_n \to 0$ (C, α) *if and only if*

$$n^{\beta+1} \sum_{\nu=n}^{\infty} s_\nu \nu^{-\beta-2} \to 0 \quad (C, \alpha-1).$$

Necessity. We can write

$$\frac{1}{m^{\beta+2}} = \sum_{\mu=0}^{k} c_\mu \frac{1}{E_m^{\beta+2+\mu}} + O(m^{-k-\beta-3}),$$

and, since $s_n = o(n^{\beta+2})$, we can choose k so large that

$$\sum_{m=n}^{\infty} s_m m^{-\beta-2} = \sum_{\mu=0}^{k} c_\mu \sum_{m=n}^{\infty} \frac{s_m}{E_m^{\beta+2+\mu}} + o(n^{-\beta-1}).$$

It is therefore sufficient to show that

$$n^{\beta+1} \sum_{m=n}^{\infty} \frac{s_m}{E_m^{\beta+2+\mu}} = o(1) \quad (C, \alpha-1) \quad (\mu = 0, 1, \dots, k).$$

Writing S_n^r for the nth (C, r) sum of $\sum (s_n - s_{n-1})$ we have, on summation by parts,

$$n^{\beta+1} \sum_{m=n}^{\infty} \frac{s_m}{E_m^{\beta+2+\mu}} = c n^{\beta+1} \sum_{m=n}^{\infty} S_m^{[\alpha]+1}/E_m^{\beta+\mu+[\alpha]+3} +$$
$$+ c_0 S_{n-1}^{[\alpha]+1} \frac{n^{\beta+1}}{E_n^{\beta+\mu+[\alpha]+2}} + \sum_{r=1}^{[\alpha]} c_r S_{n-1}^r \frac{n^{\beta+1}}{E_n^{\beta+\mu+r+1}}. \quad (5.1)$$

Since $S_{n-1}^{[\alpha]+1} = o(n^{[\alpha]+1})$, the first two terms on the right are $o(1)$. Moreover, since $S_{n-1}^r = o(n^\alpha)$, we have

$$S_{n-1}^r \frac{n^{\beta+1}}{E_n^{\beta+\mu+r+1}} = \sum_{\nu=0}^{h} c_\nu \frac{n^{-r} S_{n-1}^r}{E_n^{\mu+\nu}} + o(1) \quad (h > \alpha - r),$$

so that, since $\qquad n^{-r} S_{n-1}^r = o(1) \quad (C, \alpha - r),$

it follows [cf. (5) Lemma 3] that each member of the summation term in (5.1) is $o(1)$ $(C, \alpha-1)$.

Sufficiency. Let $\qquad x_n = \sum_{\nu=n}^{\infty} s_\nu \nu^{-\beta-2}.$

Then a simple calculation shows that

$$\sum_{\nu=1}^{n} s_\nu = -\sum_{\nu=1}^{n} x_\nu \nu^{-\beta-1} \Delta\{(\nu-1)^{\beta+2}\} - \frac{n^{\beta+2}}{(n+1)^{\beta+1}} x_{n+1},$$

† The case $\alpha = 1$ of this lemma is due to Hardy (7). The sufficiency part of the proof given here is due, essentially, to a referee, to whom I am indebted for several valuable remarks.

and hence that

$$n^{-1}\sum_{\nu=1}^{n} s_\nu = \sum_{r=0}^{k}\Big\{c_r x_n n^{-r}+n^{-1}\,d_r\sum_{\nu=1}^{n} x_\nu \nu^{-r}\Big\}+n^{-1}\sum_{\nu=1}^{n} x_\nu\, O(\nu^{-k})+ \\ +O(n^{-k+1})x_{n+1}. \quad (5.2)$$

Since $x_n \to 0\,(C,\alpha-1)$, it follows, *a fortiori*, that $x_n n^{-r}$ and $n^{-1}\sum_{\nu=1}^{n} x_\nu \nu^{-r}$ are $o(1)$ $(C,\alpha-1)$ for $r = 0, 1,..., k$. Moreover, since $x_n = o(n^{\beta+1})$, we may choose k so large that $x_n n^{-k+1} = o(1)$. Hence each term on the right of (5.2) is $o(1)$ $(C,\alpha-1)$. Consequently

$$n^{-1}\sum_{\nu=1}^{n} s_\nu = o(1) \quad (C,\alpha-1),$$

and it follows that

$$s_n = o(1) \quad (C,\alpha).$$

I remark finally that Lemma 9 also holds if $n^{\beta+1}\sum_{\nu=n}^{\infty} s_\nu \nu^{-\beta-2}$ is replaced by $n^{\beta+1}\sum_{\nu=n+1}^{\infty} s_\nu \nu^{-\beta-2}$.

Lemma 10. *If $\beta \geqslant 0$, $0 \leqslant \gamma \leqslant 1$, and if $\sum_{n=1}^{\infty} s_n n^{-\beta-2}$ is convergent, then $x^{\beta+1}S_\beta(x) = o(1)$ (C,γ) if and only if*

$$n^{\beta+1}\sum_{\nu=n+1}^{\infty} s_\nu \nu^{-\beta-2} = o(1) \quad (C,\gamma).$$

The case $\gamma = 0$ is trivial, and we suppose that $0 < \gamma \leqslant 1$.

Let

$$t_n = \sum_{\nu=n+1}^{\infty} s_\nu \nu^{-\beta-2}$$

and define $g(x)$ to be such that $g(x) = n^{\beta+1}t_n$ for $n < x \leqslant n+1$. Then,

$$n^{\beta+1}t_n = o(1) \quad (C,\gamma)$$

if and only if

$$I(\omega) = \int_1^{\omega} (\omega-x)^{\gamma-1}g(x)\,dx = o(\omega^\gamma),$$

while

$$x^{\beta+1}S_\beta(x) = o(1) \quad (C,\gamma)$$

if and only if

$$J(\omega) = \int_1^{\omega} (\omega-x)^{\gamma-1}x^{\beta+1}S_\beta(x)\,dx = o(\omega^\gamma),$$

and it is sufficient to show that $I(\omega)-J(\omega) = o(\omega^\gamma)$.

Writing $N = [\omega]-1$, we have

$$I(\omega) = \sum_{r=1}^{N-1} r^{\beta+1}t_r \int_r^{r+1} (\omega-x)^{\gamma-1}\,dx+N^{\beta+1}t_N\int_N^{\omega}(\omega-x)^{\gamma-1}\,dx, \quad (5.3)$$

$$J(\omega) = \sum_{r=1}^{N-1} t_r \int_r^{r+1} (\omega-x)^{\gamma-1}x^{\beta+1}\,dx+t_N\int_N^{\omega}(\omega-x)^{\gamma-1}x^{\beta+1}\,dx. \quad (5.4)$$

Since $n^{\beta+1}t_n = o(n^\gamma)$, it easily follows that the final term on the right of each of (5.3) and (5.4) is $o(\omega^\gamma)$. Next, since

$$\beta+1\int_r^{r+1}(\omega-x)^{\gamma-1}\,dx-\int_r^{r+1}(\omega-x)^{\gamma-1}x^{\beta+1}\,dx$$

$$=\int_r^{r+1}(\omega-x)^{\gamma-1}\{r^{\beta+1}-(r+y)^{\beta+1}\}\,dx \quad (0\leqslant y\leqslant 1)$$

$$=O\Big(r^\beta\int_r^{r+1}(\omega-x)^{\gamma-1}\,dx\Big)=O\{r^\beta(N-r)^{\gamma-1}\},$$

uniformly in r and N, it follows that the difference between the sums on the right of (5.3) and (5.4) is

$$O\Big\{\sum_{r=1}^{N-1}(N-r)^{\gamma-1}r^\beta|t_r|\Big\}=\sum_{r=1}^{N-1}(N-r)^{\gamma-1}o(r^{\gamma-1})$$
$$=o(N^{2\gamma-1})=o(\omega^\gamma)$$

since $0<\gamma\leqslant 1$.

Proof of Theorem 4. We suppose throughout that $s=0$, and we take $\beta>\alpha+\gamma$. It is sufficient to prove that $\bar{t}^{\gamma,\delta}(s_n)=o(1)$ for any convenient value of δ; the theorem then follows from Lemma 5 and Theorems 2 and 3.

Since $s_n=o(1)$ (C,α), it follows from Lemma 9 that

$$n^{\beta+1}\sum_{\nu=n}^{\infty}s_\nu\,\nu^{-\beta-2}=o(1)\quad(C,\alpha-1).$$

If, $\alpha>2$, then, since
$$\sum_{n=1}^{\infty}\sum_{\nu=n}^{\infty}s_\nu\,\nu^{-\beta-2}$$

is absolutely convergent, from a second application of Lemma 9 it follows that

$$n^\beta\sum_{m=n}^{\infty}\sum_{\nu=m}^{\infty}s_\nu\,\nu^{-\beta-2}=o(1)\quad(C,\alpha-2)$$

or, inverting the order of summation (as we may by absolute convergence), we get that

$$n^\beta\sum_{\nu=n}^{\infty}E'_{\nu-n}\,s_\nu\,\nu^{-\beta-1}=o(1)\quad(C,\alpha-2).$$

Repeating this argument p $(\leqslant\alpha)$ times we see that $s_n=o(1)$ (C,α) implies

$$n^{\beta+2-p}\sum_{\nu=n}^{\infty}E^{p-1}_{\nu-n}s_\nu\,\nu^{-\beta-1}=o(1)\quad(C,\alpha-p).\tag{5.5}$$

If α is an integer, putting $p=\alpha$ in (5.5) shows that $\bar{t}^{\alpha,\beta-\alpha+1}(s_n)=o(1)$.

In the case when α is not an integer, we observe first that it is sufficient to prove the result when $1 < \alpha < 2$. For, if $\alpha > 2$, we have

$$\bar{t}^{\gamma-[\alpha]+1,\beta-\gamma+1}(\bar{t}^{[\alpha]-1,\beta-[\alpha]+2}(s_n)) = cn^{\beta-\gamma+2}\sum_{\nu=n}^{\infty} E_{\nu-n}^{\gamma-[\alpha]}\sum_{p=\nu}^{\infty} E_{p-\nu}^{[\alpha]-2} s_p p^{-\beta-2}$$

$$= cn^{\beta-\gamma+2}\sum_{p=n}^{\infty} E_{p-n}^{\gamma-1} s_p p^{-\beta-2} = c\bar{t}^{\gamma,\beta-\gamma+1}(s_n), \tag{5.6}$$

where the inversion is justified by absolute convergence since $\beta > \alpha+\gamma$. Putting $p = [\alpha]-1$ in (5.5) shows that $\bar{t}^{[\alpha]-1,\beta-[\alpha]+2}(s_n)$ is summable $(C, \alpha-[\alpha]+1)$ to zero. Consequently, since $1 < \alpha-[\alpha]+1 < 2$ and (if $\gamma > \alpha$) $\gamma-[\alpha]+1 > \alpha-[\alpha]+1$, it will follow from (5.6) and the case $1 < \alpha < 2$ of the theorem that $\bar{t}^{\gamma,\beta-\gamma+1}(s_n) = o(1)$, and the result follows.

Finally, to prove the theorem when $1 < \alpha < 2$ it is convenient to work in terms of the equivalent $(R^*; \gamma, \delta)$ and (R, n, α) means.

Since $0 < \alpha-1 < 1$, it follows from Lemmas 9 and 10 that

$$x^{\beta+1}S_\beta(x) = o(1) \quad (C, \alpha-1).$$

Hence, from the relation

$$S_\omega^{\gamma,\beta}(s) = c\omega^{\beta+2}\int\limits_\omega^\infty \left(1-\frac{\omega}{x}\right)^{\gamma-2} x^{\beta+1}S_\beta(x)\frac{dx}{x^{\beta+3}},$$

we obtain, as in the proof of Theorem 2, using in place of Lemma 1 the result that,† for $0 < \omega \leqslant x \leqslant 2\omega$,

$$\int\limits_\omega^x t^{\beta+1}S_\beta(t)\,dt = o\{\omega^\alpha(x-\omega)^{[\alpha]-\alpha+1}\},$$

the fact that $S_\omega^{\alpha,\beta}(s) = o(1)$, and the theorem follows.

Proof of Theorem 5. We suppose that $s = 0$ and that $\beta > \alpha+\gamma$. Our hypothesis implies $\bar{t}^{\alpha,\beta}(s_n) = o(1)$.

By the arguments leading to (5.6),

$$\bar{t}^{\alpha,\beta}(s_n) = \bar{t}^{1,\beta}(\bar{t}^{\alpha-1,\beta+1}(s_n))$$

$$= cn^{\beta+1}\sum_{\nu=n}^{\infty} \bar{t}^{\alpha-1,\beta+1}(s_\nu)\nu^{-\beta-2}. \tag{5.7}$$

Since $\bar{t}^{\alpha,\beta}(s_n) = o(1)$, it follows from Lemma 9 that

$$\bar{t}^{\alpha-1,\beta+1}(s_n) = o(1) \quad (C, 1).$$

† See (8) 683.

Repeating this argument p ($\leqslant \alpha-1$) times we see that $\bar{t}^{\alpha,\beta}(s_n) = o(1)$ implies
$$\bar{t}^{\alpha-p,\beta+p}(s_n) = o(1) \quad (C,p). \tag{5.8}$$

If α is an integer, putting $p = \alpha-1$ in (5.8) shows that
$$n^{\beta+\alpha}\sum_{\nu=n}^{\infty} s_\nu \nu^{-\beta-\alpha-1} = o(1) \quad (C,\alpha-1),$$
and hence by Lemma 9 that $s_n = o(1)$ (C,α).

When α is not an integer, we again observe that it is sufficient to prove the case $1 < \alpha < 2$. For, if $\alpha > 2$, then, by (5.6) (with $\gamma = \alpha$),
$$\bar{t}^{\alpha-[\alpha]+1,\beta-\alpha+1}(\bar{t}^{[\alpha]-1,\beta-[\alpha]+2}(s_n)) = \bar{t}^{\alpha,\beta-\alpha+1}(s_n) = o(1),$$
and the case $1 < \alpha < 2$ of the theorem implies that
$$\bar{t}^{[\alpha]-1,\beta-[\alpha]+2}(s_n) = o(1) \quad (C,\alpha-[\alpha]+1+\epsilon) \quad (\epsilon > 0). \tag{5.9}$$

If $[\alpha] = 2$, then (5.9) becomes
$$n^{\beta+1}\sum_{\nu=n}^{\infty} s_\nu \nu^{-\beta-2} = o(1) \quad (C,\alpha-1+\epsilon), \tag{5.10}$$
and it follows from Lemma 9 that
$$s_n = o(1) \quad (C,\alpha+\epsilon).$$
If $[\alpha] > 2$, then successive applications of (5.7) (with β replaced by $\beta-[\alpha]+2$, and α successively by $[\alpha]-1, [\alpha]-2,\ldots$) and Lemma 9 reduce (5.9) to (5.10) and yield the required result.

Finally, to prove the theorem when $1 < \alpha < 2$ we work in terms of the equivalent $(R^*;\alpha,\delta)$ and (R,n,γ) means.

By Lemmas 9 and 10 it is sufficient to prove that
$$x^{\beta+1}S_\beta(x) = o(1) \quad (C,\gamma-1) \quad (\gamma > \alpha).$$
Now
$$\int_0^\omega (\omega-x)^{\gamma-2}x^{\beta+1}S_\beta(x)\,dx = \int_0^\omega (\omega-x)^{\gamma-2}x^{\beta+\alpha+1}T_{\alpha,\beta}(x)\,dx,$$
and the result can be obtained, as in the proof of Theorem 2, by application of Lemma 1.

REFERENCES

1. A. F. Andersen, 'Summation af ikke hel Orden', *Mat. Tidsskrift*, B 1946, 33–52.
2. D. Borwein, 'On some methods of summability', *Quart. J. of Math.* (Oxford) 9 (1958) 310.

3. L. S. Bosanquet, 'On Abel's integral equation and fractional integrals', *Proc. London Math. Soc.* (2) 31 (1930) 135–43.
4. —— 'On a theorem of M. Riesz', ibid. (3) 1 (1951) 453.
5. —— 'An extension of a theorem of Andersen', *J. London Math. Soc.* 25 (1950) 76.
6. J. J. Gergen, 'Summability of double Fourier series', *Duke Math. J.* 3 (1937) 133–48.
7. G. H. Hardy, 'A theorem concerning summable series', *Proc. Cambridge Phil. Soc.* 20 (1921) 304–7.
8. —— and J. E. Littlewood, 'Notes on the theory of series (III)', *Proc. Cambridge Phil. Soc.* 23 (1927) 681–4.
9. B. Kuttner, 'On quasi-Hausdorff transformations', *Quart. J. of Math.* (Oxford) (2) 8 (1957) 272–8.

ON CESÀRO AND QUASI-CESÀRO SUMMABILITY OF FOURIER SERIES

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1. Introduction

SUPPOSE that $\phi(t) \in L(0,\pi)$ and is even† and periodic, with period 2π, and that

$$\phi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt.$$

Write

$$\phi_0(t) = \phi(t),$$

$$\phi_h(t) = \frac{h}{t^h} \int_0^t (t-u)^{h-1} \phi(u)\, du \quad (t > 0,\ h > 0).$$

Throughout what follows $S[g]$ denotes the Fourier series of $g(t)$ at $t = 0$, or, if $g(t)$ is not periodic, of the even function which coincides with $g(t)$ in $(0,\pi)$.

The following theorem, due to Bosanquet [(**2**) Theorem 2] relates the summability (C) of given order of $S[\phi]$ to that of $S[\phi_h]$:

THEOREM A. *In order that $S[\phi]$ be summable (C,α) to s it is necessary and sufficient that $S[\phi_h]$ be summable $(C,\alpha-h)$ to s, where h is an integer such that $0 \leqslant h \leqslant \alpha$.*

Bosanquet remarks‡ [(**2**) 17, footnote ‡] that there is a possibility of completing Theorem A (i.e. of removing the restriction that h be an integer) if the functional mean $\phi_h(t)$ is replaced by some other definition. The object of this note is to show that a 'right' definition is the analogue for functions of the quasi-Cesàro means recently studied by Kuttner (**11**). In fact, writing

$$\phi_0^*(t) = \phi(t),$$

$$\phi_h^*(t) = ht \int_t^{\infty} (u-t)^{h-1} \phi(u) \frac{du}{u^{h+1}} \quad (t > 0,\ h > 0), \tag{1.1}$$

† There is no loss of generality, in this note, in considering even functions whose Fourier series has no constant term.

‡ I am indebted to Dr. Bosanquet for showing me some correspondence with the late Professor Hardy on this subject.

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where the integral on the right of (1.1) is absolutely convergent since $|\phi(u)|$ is periodic, and consistency follows from (3.3), we prove the theorem:

THEOREM 1. *In order that $S[\phi]$ be summable (C,α) to s it is necessary and sufficient that $S[\phi_h^*]$ be summable $(C,\alpha-h)$ to s, where $0 \leqslant h < \alpha$.*

I also show that the mean $\phi_h(t)$ used in Theorem A is the 'right' one for the solution of the corresponding problem with quasi-Cesàro summability in place of Cesàro summability, in the theorem:

THEOREM 2. *In order that $S[\phi]$ be summable (C^*,α) to s it is necessary and sufficient that $S[\phi_h]$ be summable $(C^*,\alpha-h)$ to s, where $0 \leqslant h < \alpha$.*

In what follows I assume (particularly in § 4) an acquaintance with the definitions and main results of (**13**) and give some further notation and some preliminaries in § 2. The proofs of Theorems 1 and 2 occupy §§ 3 and 4 respectively, and I conclude with some remarks concerning the relation between my results and Theorem A.

2. Throughout this note c, c_0, c_1,... denote constants, not necessarily the same at each occurrence, and $\sum$ denotes $\sum\limits_1^\infty$.

If $\alpha \geqslant 0$ and $\phi_\alpha^*(t) \to 0$ $(\phi_\alpha(t) \to 0)$ as $t \to +0$, we write $\phi(t) \to 0$ (C^*,α) $((C,\alpha))$ as $t \to +0$.

We shall require the functions $\gamma_\alpha(\beta,t)$ and $\gamma_\alpha^*(\beta,t)$, defined for $\alpha > 0$, by

$$\gamma_\alpha(\beta,t) = \int\limits_0^1 (1-u)^\alpha u^\beta \cos ut\, du \quad (\beta \geqslant 0), \tag{2.1}$$

$$\gamma_\alpha^*(\beta,t) = \int\limits_1^\infty \left(1-\frac{1}{u}\right)^\alpha \cos ut \frac{du}{u^{\beta+1}} \quad (\beta > 0). \tag{2.2}$$

From the Fourier-integral inversion formulae [(**12**) Theorem 3] we have, for $\alpha > 0$,

$$\int\limits_0^\infty \gamma_\alpha^*(\beta,t)\cos ut\, dt = \begin{cases} cu^{-\beta-1}\{1-(1/u)\}^\alpha & (u \geqslant 1) \\ 0 & (0 \leqslant u < 1) \end{cases} \quad (\beta > 0), \tag{2.3}$$

$$\int\limits_0^\infty \gamma_\alpha(\beta,t)\cos ut\, dt = \begin{cases} cu^\beta(1-u)^\alpha & (0 \leqslant u \leqslant 1) \\ 0 & (u > 1) \end{cases} \quad (\beta \geqslant 0). \tag{2.4}$$

It can be verified that, if $\alpha > h > 0$,

$$\gamma^*_\alpha(\beta,t) = c\int\limits_1^\infty (u-1)^{h-1}\gamma^*_{\alpha-h}(\beta+h,ut)\frac{du}{u^{\beta+h}} \quad (\beta > 0), \tag{2.5}$$

$$\gamma_\alpha(\beta,t) = c\int\limits_0^1 (1-u)^{h-1}u^{\beta+\alpha+1-h}\gamma_{\alpha-h}(\beta,ut)\,du \quad (\beta \geqslant 0). \tag{2.6}$$

Clearly $\gamma^*_\alpha(\beta,t)$ and $\gamma_\alpha(\beta,t)$ are bounded for all t, and it can be shown by standard methods [for (2.7) see (**2**) 19; cf. also (**6**)] that, for large t,

$$\gamma^*_\alpha(\beta,t) = \frac{c\sin(t+\frac{1}{2}\pi\alpha)}{t^{\alpha+1}}+O(t^{-\alpha-2}) \quad (\beta > 0), \tag{2.7}$$

$$\gamma_\alpha(\beta,t) = \frac{c\sin\frac{1}{2}\pi\beta}{t^{\beta+1}}+\frac{c_1\sin(t+\frac{1}{2}\pi\alpha)}{t^{\alpha+1}}+O(t^{-\alpha-2})+O(t^{-\beta-2}) \quad (\beta \geqslant 0), \tag{2.8}$$

and these formulae may be differentiated.

If $S[g] = \sum u_n$, we write

$$S_\omega(\alpha,\beta,g) = \sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^\alpha\left(\frac{n}{\omega}\right)^\beta u_n, \tag{2.9}$$

$$S^*_\omega(\alpha,\beta,g) = \sum_{n>\omega}\left(1-\frac{\omega}{n}\right)^\alpha\left(\frac{\omega}{n}\right)^{\beta+1} u_n, \tag{2.10}$$

subject to the convergence of the series on the right of (2.10).

It can be verified† that $S_\omega(\alpha,\beta,g) \to s$ as $\omega \to \infty$ if and only if

$$\binom{n+\alpha+\beta}{\alpha+\beta}^{-1}\sum_{m=0}^{n}\binom{n-m+\alpha}{\alpha}\binom{m+\beta}{\beta}u_m \to s, \tag{2.11}$$

as $n \to \infty$. If $\sum u_n$ is summable (C), then $S_\omega(\alpha,\beta,g) \to s$ if and only if $\sum u_n = s\ (C,\alpha)$. For, supposing (as we may) that $s = 0$, (2.11) becomes

$$\binom{n+\beta}{\beta}u_n = o(n^{\beta-1}) \quad (C,\alpha+1),$$

which [(**4**) 76] is equivalent to

$$nu_n = o(1) \quad (C,\alpha+1).$$

Thus, if $\sum u_n = 0\ (C,\alpha)$, (2.11) holds and $S_\omega(\alpha,\beta,g) \to 0$. Conversely, if $S_\omega(\alpha,\beta,g) \to 0$, (2.11) holds, and, if $\sum u_n = 0\ (C)$, this implies [(**2**) Lemma 6] that $\sum u_n = 0\ (C,\alpha)$.

Finally, we remark (**13**) that $S^*_\omega(\alpha,\beta,g)$ is the $(R^*;\alpha+1,\beta)$ mean,

† The case $\beta = 0$ is well known, being the equivalence of summability (C,α) and (R,n,α), a compact proof of which is given by Gergen (**7**). The result stated above may be obtained by a straightforward adaptation of Gergen's analysis.

of rank ω, of the sequence $\{nu_n\}$, so that $S^*_\omega(\alpha,\beta,g) \to 0$ if and only if $\{nu_n\}$ is summable $(R^*;\, \alpha+1,\beta)$ to zero.

3. In this section we prove Theorem 1. We require four lemmas:

LEMMA 3.1. *If $f(t) \in L(0,\pi)$, then $S[f]$ is summable (C) to s if and only if $f(t)+f(-t) \to 2s$ (C) as $t \to +0$.*

See Hardy and Littlewood (**8**).

LEMMA 3.2. (i) *If $\phi_\alpha(t) = o(1)$ as $t \to +0$, then $\phi^*_\gamma(t) = o(1)$ as $t \to +0$, where $\gamma \geqslant \alpha > 0$ if α is an integer and $\gamma > \alpha > 0$ otherwise.*

(ii) *If $\phi^*_\alpha(t) = o(1)$ as $t \to +0$, then $\phi_\gamma(t) = o(1)$ as $t \to +0$ where $\gamma \geqslant \alpha > 0$ if α is an integer and $\gamma > \alpha > 0$ otherwise.*

(This is proved by standard methods; I give an outline of the details for convenience.)

Proof of (i). In view of the periodicity of $\phi(t)$ we have, for $r = 1, 2,\ldots,$

$$\phi_r(t) = O(t^{-1}) \quad \text{as } t \to \infty. \tag{3.1}$$

Integrating by parts† we get

$$\phi^*_\gamma(t) = t\sum_{r=1}^{[\alpha]} \left[c_r u^r \phi_r(u)\left(\frac{d}{du}\right)^{r-1}\{(u-t)^{\gamma-1}u^{-\gamma-1}\}\right]_t^\infty +$$

$$+ct\int\limits_t^\infty u^{[\alpha]}\phi_{[\alpha]}(u)\left(\frac{d}{du}\right)^{[\alpha]}\{(u-t)^{\gamma-1}u^{-\gamma-1}\}\,du. \tag{3.2}$$

If α is an integer, then from (3.1) and $\phi_\alpha(t) = o(1)$, (3.2) (with $\gamma = \alpha$) gives

$$\phi^*_\alpha(t) = c_1\phi_\alpha(t)+c_2 t\int\limits_t^\infty u^\alpha\phi_\alpha(u)\left(\frac{d}{du}\right)^\alpha\{(u-t)^{\alpha-1}u^{-\alpha-1}\}\,du = o(1).$$

If α is not an integer, we take $\gamma > \alpha$ in (3.2), and use (3.1) to give

$$\phi^*_\gamma(t) = ct\int^\infty u^{[\alpha]}\phi_{[\alpha]}(u)\left(\frac{d}{du}\right)^{[\alpha]}\{(u-t)^{\gamma-1}u^{-\gamma-1}\}\,du$$

$$= ct\int\limits_t^{2t} +ct\int\limits_{2t}^\infty = J_1+J_2.$$

That each of J_1, J_2 is $o(1)$ as $t \to +0$ is shown in each case by a single

† If $[\alpha] = 0$ the summation term is absent.

integration by parts and use of the facts [for the first of which see (9) 683] that

$$\int_t^x u^{[\alpha]}\phi_{[\alpha]}(u)\,du = o\{t^\alpha(x-t)^{[\alpha]-\alpha+1}\} \quad (0 < t \leqslant x \leqslant 2t),$$

$$\phi_{[\alpha]+1}(t) = o(1).$$

Proof of (ii). It can be verified that, for $\alpha > \beta \geqslant 0$,

$$\phi^*_\alpha(x) = cx^{\beta+1}\int_x^\infty \left(1-\frac{x}{t}\right)^{\alpha-\beta-1}\phi^*_\beta(t)\frac{dt}{t^{\beta+2}}, \tag{3.3}$$

and that, since $t\phi_1(t) = o(1)$ as $t \to +0$,

$$\phi^*_r(x) = o(x^{-1}) \quad \text{as } x \to +0 \quad (r = 1, 2, \ldots). \tag{3.4}$$

Integrating by parts we have

$$x^\gamma\phi_\gamma(x) = \sum_{r=1}^{[\alpha]}\left[c_r\,t^{-r}\phi^*_r(t)\left(t^2\frac{d}{dt}\right)^{r-1}\{(x-t)^{\gamma-1}t^2\}\right]_0^x +$$

$$+c\int_0^x t^{-[\alpha]-2}\phi^*_{[\alpha]}(t)\left(t^2\frac{d}{dt}\right)^{[\alpha]}\{(x-t)^{\gamma-1}t^2\}\,dt. \tag{3.5}$$

If α is an integer, putting $\gamma = \alpha$ in (3.5) and using (3.4), we have

$$x^\alpha\phi_\alpha(x) = cx^\alpha\phi^*_\alpha(x)+c_1\int_0^x t^{-\alpha-2}\phi^*_\alpha(t)\left(t^2\frac{d}{dt}\right)^\alpha\{(x-t)^{\alpha-1}t^2\}\,dt = o(x^\alpha)$$

since $\phi^*_\alpha(t) = o(1)$.

If α is not an integer, then taking $\gamma > \alpha$ in (3.5) and using (3.4) we have

$$x^\gamma\phi_\gamma(x) = c\int_0^x t^{-[\alpha]-2}\phi^*_{[\alpha]}(t)\left(t^2\frac{d}{dt}\right)^{[\alpha]}\{(x-t)^{\gamma-1}t^2\}\,dt$$

$$= c\int_0^{\frac{1}{2}x} + c\int_{\frac{1}{2}x}^x = I_1+I_2.$$

That $I_1 = o(x^\gamma)$ follows on integrating by parts and using the result that

$$\int_x^\infty t^{-[\alpha]-2}\phi^*_{[\alpha]}(t)\,dt = x^{-[\alpha]-1}\phi^*_{[\alpha]+1}(x) = o(x^{-[\alpha]-1}).$$

By (3.3),

$$\int_x^\infty \left(1-\frac{x}{t}\right)^{[\alpha]-\alpha} \phi_\alpha^*(t)\frac{dt}{t^{\alpha+1}}$$

$$= c\int_x^\infty \left(1-\frac{x}{t}\right)^{[\alpha]-\alpha}\frac{dt}{t^{\alpha-[\alpha]}}\int_t^\infty \left(1-\frac{t}{u}\right)^{\alpha-[\alpha]-1}\phi_{[\alpha]}^*(u)\frac{du}{u^{[\alpha]+2}}$$

$$= c\int_x^\infty \phi_{[\alpha]}^*(u)\frac{du}{u^{\alpha+1}}\int_x^u (t-x)^{[\alpha]-\alpha}(u-t)^{\alpha-[\alpha]-1}\,dt$$

$$= c\int_x^\infty \phi_{[\alpha]}^*(u)\frac{du}{u^{\alpha+1}},$$

where the inversion is justified by absolute convergence in view of the periodicity of $\phi(u)$. Consequently, by a standard method [(9) 683] we have, for $0 < \frac{1}{2}x \leqslant t \leqslant x$,

$$c\int_t^x \phi_{[\alpha]}^*(u)\frac{du}{u^{\alpha+1}} = \int_t^\infty \left(1-\frac{t}{u}\right)^{[\alpha]-\alpha}\phi_\alpha^*(u)\frac{du}{u^{\alpha+1}} - \int_x^\infty \left(1-\frac{x}{u}\right)^{[\alpha]-\alpha}\phi_\alpha^*(u)\frac{du}{u^{\alpha+1}}$$

$$= o\{t^{-[\alpha]-1}(x-t)^{[\alpha]-\alpha+1}\}.$$

Use of this fact after a single integration by parts shows that $I_2 = o(x^\gamma)$ and completes the proof of (ii).

LEMMA 3.3. *If $\alpha > 0$, $\beta > 0$ and if*

$$\phi_\alpha^*(t) = o(1) \quad (C,\beta) \quad (t \to +0),$$

then $$\phi(t) = o(1) \quad (C^*;\alpha+\beta+\epsilon) \quad (t \to +0),$$

for every positive ϵ. Conversely, if

$$\phi(t) = o(1) \quad (C^*,\alpha+\beta) \quad (t \to +0),$$

then $$\phi^*(t) = o(1) \quad (C,\beta+\epsilon) \quad (t \to +0),$$

for every positive ϵ.

We have, for fixed positive x,

$$\beta\int_0^1 (1-t)^{\beta-1}\phi_\alpha^*(tx)\,dt = \beta\int_0^1 (1-t)^{\beta-1}\,dt\int_1^\infty \left(1-\frac{1}{u}\right)^{\alpha-1}\phi(utx)\frac{du}{u^2}$$

$$= \int_1^\infty \left(1-\frac{1}{u}\right)^{\alpha-1}\frac{du}{u^2}\beta\int_0^1 (1-t)^{\beta-1}\phi(utx)\,dt$$

$$= \int_1^\infty \left(1-\frac{1}{u}\right)^{\alpha-1}\phi_\beta(ux)\frac{du}{u^2}, \tag{3.6}$$

the inversion being justified by absolute convergence since ($|\phi(t)|$ being periodic)

$$\int_0^1 (1-t)^{\beta-1}|\phi(utx)|\,dt = O(1) \quad (C,1)$$

as $u \to \infty$. Thus $\phi_\alpha^*(t) = o(1)$ (C,β) is equivalent to $\phi_\beta(t) = o(1)$ (C^*,α) and thus by Lemma 3.2 implies $\phi_\beta(t) = o(1)$ $(C,\alpha+\epsilon)$, for every positive ϵ. By a known result [(**5**) Theorem 13] this is equivalent to

$$\phi(t) = o(1) \quad (C,\alpha+\beta+\epsilon)$$

and hence, from Lemma 3.2 again, it follows that $\phi_\alpha^*(t) = o(1)$ (C,β) implies that

$$\phi(t) = o(1) \quad (C^*,\alpha+\beta+\epsilon') \quad (\epsilon' > \epsilon > 0).$$

Conversely, if $\phi(t) = o(1)$ $(C^*;\alpha+\beta)$, then, by Lemma 3.2,

$$\phi(t) = o(1) \quad (C,\alpha+\beta+\epsilon),$$

i.e. $$\phi_{\beta+\epsilon-\epsilon'}(t) = o(1) \quad (C,\alpha-\epsilon') \quad (0 < \epsilon' < \max(\epsilon,\alpha)).$$

Using Lemma 3.2 again we get

$$\phi_{\beta+\epsilon-\epsilon'}(t) \to 0 \quad (C^*,\alpha)$$

and so from (3.6) (with β replaced by $\beta+\epsilon-\epsilon'$)

$$\phi_\alpha^*(t) \to 0 \quad (C,\beta+\epsilon-\epsilon').$$

LEMMA 3.4. *If $\beta \geqslant 0$, $\alpha > 0$; if $g(t) \in L(0,\pi)$ and if*

$$\int_\pi^\infty |g(t)|\frac{dt}{t^{1+\epsilon}} < \infty,$$

for arbitrary positive ϵ, then, for $\omega > 0$,

$$S_\omega(\alpha,\beta,g) = c\omega\int_0^\infty g(t)\gamma_\alpha(\beta,\omega t)\,dt + o(1).$$

The error term is not required if $g(t)$ is periodic. If $\beta = 0$, this result is well known [see (**10**) 567].

Let $\bar{g}(t) = g(t)$ in $0 < t < \pi$ and be even and periodic with period 2π. A standard argument [(**10**) 567] shows that

$$S_\omega(\alpha,\beta,g) = c\omega\int_0^\infty \bar{g}(t)\gamma_\alpha(\beta,\omega t)\,dt.$$

Since $|\tilde{g}(t)|$ is periodic,

$$\int_{\pi}^{\infty} |\tilde{g}(t)| \frac{dt}{t^{1+\alpha}} < \infty,$$

so that, from (2.8),

$$\omega \int_{\pi}^{\infty} |\tilde{g}(t)|\, |\gamma_{\alpha}(\beta, \omega t)|\, dt = o(1),$$

$$\omega \int_{\pi}^{\infty} |g(t)|\, |\gamma_{\alpha}(\beta, \omega t)|\, dt = o(1),$$

as $\omega \to \infty$, and the result stated follows immediately.

Proof of Theorem 1. We may suppose throughout that $s = 0$.

If $\alpha > h > 0$, $r \geqslant 0$, and if $\beta \geqslant 0$, then, by Lemma 3.4 and (2.6),

$$\begin{aligned} S_{\omega}(\alpha+r, \beta, \phi) &= c\omega \int_{0}^{\infty} \phi(t)\gamma_{\alpha+r}(\beta, \omega t)\, dt \\ &= c\omega \int_{0}^{\infty} \phi(t)\frac{dt}{t} \int_{0}^{t} \left(1-\frac{u}{t}\right)^{h+r-1} \left(\frac{u}{t}\right)^{\alpha+\beta+1-h} \gamma_{\alpha-h}(\beta, \omega u)\, du \\ &= c\omega \int_{0}^{\infty} \gamma_{\alpha-h}(\beta, \omega u) u^{\alpha+\beta+1-h}\, du \int_{u}^{\infty} (t-u)^{h+r-1} t^{-\alpha-\beta-r-1}\phi(t)\, dt, \end{aligned} \tag{3.7}$$

where the inversion is justified by absolute convergence since, if ω is fixed, then, by (2.8),

$$\int_{0}^{t} \left(1-\frac{u}{t}\right)^{h+r-1} \left(\frac{u}{t}\right)^{\alpha+\beta+1-h} |\gamma_{\alpha-h}(\beta, \omega u)|\, du \leqslant c \min(t, t^{-\alpha}, t^{-\beta}).$$

Write, for $p \geqslant 0$, $h > 0$, $u > 0$,

$$\psi_{h}^{p}(u) = \int_{1}^{\infty} (t-1)^{h-1} t^{-h-1-p} \phi(tu)\, dt, \tag{3.8}$$

and let $q = [\alpha]+1$. Choose β $(= \beta_0)$, so that $\alpha+\beta_0-h = q$. We then have, from (3.7) and Lemma 3.4,

$$\begin{aligned} S_{\omega}(\alpha+r, \beta_0, \phi) &= c\omega \int_{0}^{\infty} \psi_{h+r}^{q}(u)\gamma_{\alpha-h}(\beta_0, \omega u)\, du \\ &= cS_{\omega}(\alpha-h, \beta_0, \psi_{h+r}^{q})+o(1) \end{aligned} \tag{3.9}$$

since from (3.8), if $\epsilon > 0$,

$$\int_{\pi}^{\infty} |\psi_{h+r}^{q}(u)| u^{-1-\epsilon}\, du \leqslant \int_{\pi}^{\infty} u^{-\epsilon}\, du \int_{u}^{\infty} \left(1-\frac{u}{y}\right)^{h+r-1} |\phi(y)| \frac{dy}{y^2}$$

$$= \int_{\pi}^{\infty} |\phi(y)| \frac{dy}{y^2} \int_{\pi}^{y} \left(1-\frac{u}{y}\right)^{h+r-1} u^{-\epsilon}\, du$$

$$\leqslant \int_{\pi}^{\infty} |\phi(y)| \frac{dy}{y^{1+\epsilon}} < \infty,$$

because of the periodicity of $|\phi(y)|$.

Proof of necessity. If $S[\phi]$ is summable (C, α) to zero, then [cf. the remarks following (2.11)], by (3.9) (with $r = 0$), $S[\psi_h^q]$ is summable $(C, \alpha-h)$ to zero. On the other hand

$$\psi_h^q(t) = \sum_{\nu=0}^{q} (-)^{\nu} \binom{q}{\nu} (h+\nu)^{-1} \phi_{h+\nu}^{*}(t). \tag{3.10}$$

Hence, either $S[\phi_h^*]$ is summable $(C, \alpha-h)$ to zero or there is an integer ρ_1 $(1 \leqslant \rho_1 \leqslant q)$ such that $S[\phi_{h+\rho_1}^*]$ is not summable $(C, \alpha-h)$ to zero. If the first possibility holds, then there is nothing further to prove. Suppose then that the second possibility holds. By consistency $S[\phi]$ is summable $(C, \alpha+\rho_1)$ to zero and so, with $r = \rho_1$ in (3.9), $S[\psi_{h+\rho_1}^q]$ is summable $(C, \alpha-h)$ to zero. Replacing h by $h+\rho_1$ in (3.10) we see that there is an integer ρ_2 $(\rho_1 < \rho_2 \leqslant \rho_1+q)$ such that $S[\phi_{h+\rho_2}^*]$ is not summable $(C, \alpha-h)$ to zero. Proceeding in this way we obtain a sequence of integers

$$1 \leqslant \rho_1 < \rho_2 < \ldots$$

such that $S[\phi_{h+\rho_r}^*]$ is not summable $(C, \alpha-h)$ to zero for $r = 1, 2, \ldots$. By a known theorem [(3) Theorem 1] this implies that

$$\phi_{h+\rho_r}^*(t) \nrightarrow 0 \quad (C, \epsilon) \quad (0 \leqslant \epsilon < \alpha-h)$$

as $t \to +0$, and hence by Lemma (3.3) that

$$\phi(t) \nrightarrow 0 \quad (C, h+\rho_r) \quad (r = 1, 2, \ldots).$$

By consistency this implies that $\phi(t) \nrightarrow 0$ (C, τ) for any $\tau \geqslant 0$ and hence, by Lemma 3.1, that $S[\phi]$ is not summable (C) to zero. This provides a contradiction and completes the proof of necessity.

Proof of sufficiency. Suppose that $S[\phi_h^*]$ be summable $(C, \alpha-h)$ to zero. It is sufficient to show that $S[\phi_{h+\nu}^*]$ is summable $(C, \alpha-h)$ to zero for $\nu = 1, 2, \ldots$. For this implies, by (3.10) and (3.9) with $r = 0$, that

$S_\omega(\alpha, \beta_0, \phi) = o(1)$. Moreover, by Lemma 3.1, $\phi_h^*(t) = o(1)$ (C), and so, by Lemmas 3.3 and 3.2, $\phi(t) = o(1)$ (C) and hence, by Lemma 3.1 again, $S[\phi] = 0$ (C). These two facts (see the remarks following (2.11)) imply that $S[\phi] = 0$ (C, α).

Firstly $S[\phi_{h+\nu}^*]$ is summable $(C, \alpha-h)$ to zero for $\nu \geqslant 2$. For by (3) Theorem 4

$$\phi_h^*(t) \to 0 \quad (C, \alpha-h+2-\epsilon) \quad (0 < \epsilon < 1).$$

Hence, by Lemma 3.3,

$$\phi(t) \to 0 \quad (C, \alpha+2-\epsilon') \quad (0 < \epsilon' < 1).$$

Hence, by consistency,

$$\phi(t) \to 0 \quad (C, \alpha+\nu-\epsilon') \quad (\nu = 2, 3,\dots;\ 0 < \epsilon' < 1)$$

and so, by Lemma 3.3,

$$\phi_{h+\nu}^*(t) \to 0 \quad (C, \alpha-h-\epsilon'') \quad (0 < \epsilon'' < \min(\alpha-h, 1)).$$

Our assertion now follows from a known result [(3) Theorem 1].

We also have

$$\phi_{\tau+1}^*(x) = (\tau+1) \int_1^\infty (t-1)^\tau \phi(tx) \frac{dt}{t^{\tau+2}}$$

$$= (\tau+1) \int_1^\infty \phi_1(tx) t\{(\tau+2)(t-1)^\tau t^{-1} - \tau(t-1)^{\tau-1}\} \frac{dt}{t^{\tau+2}}$$

$$= (\tau+1)(\tau+2) \int_1^\infty (t-1)^\tau \phi_1(tx) \frac{dt}{t^{\tau+2}} - \tau(\tau+1) \int_1^\infty (t-1)^{\tau-1} \phi_1(tx) \frac{dt}{t^{\tau+1}}$$

$$= \frac{(\tau+2)}{x} \int_0^x \phi_{\tau+1}^*(t)\, dt - \frac{(\tau+1)}{x} \int_0^x \phi_\tau^*(t)\, dt, \tag{3.11}$$

by the arguments used in Lemma 3.3. Since $S[\phi_{h+2}^*]$ and $S[\phi_h^*]$ are summable $(C, \alpha-h)$, and *a fortiori* $(C, \alpha-h+1)$, to zero, it follows from Theorem A that the Fourier series at $x = 0$ of

$$\frac{1}{x} \int_0^x \phi_{h+2}^*(t)\, dt, \qquad \frac{1}{x} \int_0^x \phi_h^*(t)\, dt$$

are summable $(C, \alpha-h)$ to zero. Putting $\tau = h+1$ in (3.11) we see that the Fourier series at $x = 0$ of

$$\frac{1}{x} \int_0^x \phi_{h+1}^*(t)\, dt$$

is summable $(C, \alpha-h)$ to zero. Finally putting $\tau = h$ in (3.11) we see that $S[\phi^*_{h+1}]$ is summable $(C, \alpha-h)$ to zero.

4. Proof of Theorem 2

We require three more lemmas:

LEMMA 4.1. *If $u_n = o(1)$ as $n \to \infty$ and if $\sum u_n$ is summable (C), then the (C^*, α) $(\alpha > 0)$ method is applicable to the sequences $\{nu_n\}$ and $\left\{\sum_{\nu=1}^{n} u_\nu\right\}$.*

We have to show that the series $\sum \nu^{-1}u_\nu$ and $\sum \nu^{-2} \sum_{r=1} u_r$ are convergent.

Since $\sum u_n$ is summable (C), so [(**1**) 57] is $\sum n^{-1}u_n$. But, since $n^{-1}u_n = o(n^{-1})$, it follows from Tauber's theorem that $\sum n^{-1}u_n$ is convergent. The other result can now be obtained by partial summation, or by appeal to a theorem by Andersen [(**1**), as extended by Bosanquet and Chow cf. (**4**) Theorem B(ii)].

LEMMA 4.2. *If $\alpha > 0$, $\beta \geqslant 0$, then $\sum u_n$ is summable $(C^*; \alpha, \beta)$ to s if and only if*
$$nu_n = o(1) \quad (C^*; \alpha+1, \beta)$$
and $\sum u_n$ is summable (C) to s.

Suppose that $\sum u_n$ is summable $(C^*; \alpha, \beta)$ to s. Then,† by Lemma 8*,
$$nu_n = o(1) \quad (C^*; \alpha+1, \beta)$$
and $\sum u_n$ is summable $(C^*; \alpha+1, \beta)$, and hence by Theorem 5* is summable (C) to s.

Conversely, if
$$nu_n = o(1) \quad (C^*; \alpha+1, \beta),$$
then $(C^*; \alpha, \beta)$ is applicable to $\sum u_n$. Hence, by Theorem 4*, $\sum u_n$ is summable (C) to s if and only if it is summable $(C^*; \gamma, \beta)$ to s for some γ. If we choose γ so that $\gamma-\alpha$ is a positive integer, then it follows, by consistency, that
$$nu_n = o(1) \quad (C^*; \gamma, \beta)$$
and hence by Lemma 8* that $\sum u_n$ is summable $(C^*; \gamma-1, \beta)$ to s. Using this argument $\gamma-\alpha$ times we obtain the required result.

LEMMA 4.3. *If $\alpha > 0$ and $\beta > 0$, if $g(t) \in L(0, \pi)$, and if*
$$\int_{\pi}^{\infty} |g(t)| t^{-1-\rho}\, dt < \infty \quad (\rho > 0),$$
then, for $\omega > 0$,
$$S^*_\omega(\alpha, \beta, g) = c\omega \int_{0}^{\infty} g(t) \gamma^*_\alpha(\beta, \omega t)\, dt + o(1). \tag{4.1}$$

† Starred numbers refer to the lemmas and theorems of (**13**).

See the remarks on Lemma 3.4. Again the error term is not required if $g(t)$ is periodic.

Proof of Theorem 2. From Lemma 4.3 and (2.5), if $\alpha > h > 0$, $r \geqslant 0$, $\beta > 0$, then

$$\begin{aligned} S^*_\omega(\alpha+r,\beta,\phi) &= c\omega \int_0^\infty \phi(t)\gamma^*_{\alpha+r}(\beta,\omega t)\,dt \\ &= c\omega \int_0^\infty \phi(t)t^\beta\,dt \int_t^\infty (u-t)^{h+r-1}\gamma^*_{\alpha-h}(\beta+h+r,\omega u)\frac{du}{u^{\beta+h+r}} \\ &= c\omega \int_0^\infty \gamma^*_{\alpha-h}(\beta+h+r,\omega u)\frac{du}{u^{\beta+h+r}} \int_0^u (u-t)^{h+r-1}t^\beta\phi(t)\,dt, \end{aligned} \tag{4.2}$$

where the inversion is justified by absolute convergence since

$$\int_0^1 (1-t)^{h+r-1}|\phi(tu)|t^\beta\,dt = O(1) \quad (C,1),$$

as $u \to \infty$.

Putting $\beta = 1$ in (4.2) we have

$$\begin{aligned} S^*_\omega(\alpha+r,1,\phi) = c_1 S^*_\omega(\alpha-h,h+r+1,\phi_{h+r})+ \\ +c_2 S^*_\omega(\alpha-h,h+r+1,\phi_{h+r+1})+o(1), \end{aligned} \tag{4.3}$$

since it is easily verified that

$$\int_\pi^\infty |\phi_{h+r+\nu}(t)|t^{-1-\rho}\,dt < \infty \quad (\rho > 0,\ h \geqslant 0,\ r \geqslant 0,\ \nu \geqslant 0).$$

Proof of sufficiency. Suppose that $S[\phi_h]$ is summable $(C^*;\alpha-h)$ to zero. It follows from Theorem 4* that $S[\phi_h]$ is summable (C), and hence from Theorem A that $S[\phi]$ is summable (C) to zero. Consequently by Lemma 4.2 it is sufficient to show that

$$na_n \to 0 \quad (C^*,\alpha+1)$$

as $n \to \infty$. Since, by Lemma 4.1, $(C^*,\alpha+1)$ is applicable to $\{na_n\}$, it follows from Theorem 2* that it is sufficient to show that

$$na_n \to 0 \quad (C^*;\ \alpha+1,\beta)$$

for some β $(\geqslant 0)$ and hence by the equivalence of $(C^*;\ \alpha+1,\beta)$ and $(R^*;\ \alpha+1,\beta)$ that it is sufficient to show that

$$S^*_\omega(\alpha,\beta,\phi) \to 0, \tag{4.4}$$

as $\omega \to \infty$, for some β $(\geqslant 0)$.

Since $S[\phi_h]$ is summable $(C^*, \alpha-h)$ to zero, it follows from Lemma 4.2 and Theorem 2* that

$$S^*_\omega(\alpha-h, \beta, \phi_h) \to 0, \tag{4.5}$$

as $\omega \to \infty$ for any $\beta \geqslant 0$. It also follows from Theorem 5* that $S[\phi_h]$ is summable $(C, \alpha-h+1-\epsilon)$, where $0 < \epsilon < \min(\alpha-h, 1)$. Consequently, by Theorem A, $S[\phi_{h+1}]$ is summable $(C, \alpha-h-\epsilon)$ and so, by Theorem 4* and Lemma 4.1, is summable $(C^*, \alpha-h)$. By Lemma 4.2 and Theorem 2* this implies that

$$S^*_\omega(\alpha-h, \beta, \phi_{h+1}) \to 0 \tag{4.6}$$

as $\omega \to \infty$ for $\beta \geqslant 0$.

Putting $\beta = h+1$ in (4.5) and (4.6) and using them in (4.3) (with $r = 0$) we obtain (4.4) with $\beta = 1$.

Proof of necessity. Suppose that $S[\phi]$ is summable (C^*, α) to zero. It follows from Theorem 5* that $S[\phi]$ is summable (C), and hence by Theorem A that $S[\phi_h]$ is summable (C), to zero. By Lemma 4.2 it is sufficient to show that

$$na_n(h) \to 0 \quad (C^*, \alpha-h+1), \tag{4.7}$$

as $n \to \infty$, where we suppose that

$$\phi_h(t) \sim \sum a_n(h)\cos nt.$$

Suppose (4.7) false. It follows from Theorem 2* that†

$$na_n(h) \nrightarrow 0 \quad (C^*; \alpha-h+1, \beta)$$

for any $\beta \geqslant 0$ and hence, by the equivalence of (R^*) and (C^*), that

$$S^*_\omega(\alpha-h, \beta, \phi_h) \nrightarrow 0 \tag{4.8}$$

as $\omega \to \infty$. From (4.8) (with $\beta = h+1$) and (4.3) (with $r = 0$) it follows that

$$S^*_\omega(\alpha-h, h+1, \phi_{h+1}) \nrightarrow 0$$

as $\omega \to \infty$ and hence, by using Theorem 2* again, that

$$S^*_\omega(\alpha-h, h+2, \phi_{h+1}) \nrightarrow 0$$

as $\omega \to \infty$. Putting $r = 1$ in (4.3) we get that

$$S^*_\omega(\alpha-h, h+2, \phi_{h+2}) \nrightarrow 0$$

as $\omega \to \infty$, and, by repetition of this argument, that generally

$$S^*_\omega(\alpha-h, h+p, \phi_{h+p}) \nrightarrow 0$$

as $\omega \to \infty$, for $p = 0, 1, 2,\ldots$. Hence, by Lemma 4.2, $S[\phi_{h+p}]$ is not summable $(C^*; \alpha-h, h+p)$ for $p = 0, 1, \ldots$ and so by Theorem 5* it

† Since $S[\phi_h]$ is summable (C), it follows from Lemma 4.1 that $(C^*, \alpha-h)$ is applicable to $S[\phi_h]$, $S[\phi_{h+1}]$,... .

is not summable (C,ϵ) $(0<\epsilon<\alpha-h)$. By arguments used in the necessity part of Theorem 1 this implies that $S[\phi]$ is not summable (C). This, by Theorem 4*, implies that $S[\phi]$ is not summable $(C^*,\alpha-h)$, which provides a contradiction. Hence (4.6) holds, and the proof of Theorem 2 is complete.

5. I conclude by remarking on the connexion between Theorems 1 and 2 and Theorem A.

Suppose that h is a positive integer, that $\alpha>h$, and that $S[\phi]$ is summable (C,α) to zero. It follows from Lemma 3.1 that $\phi_{h+p}(t)\to 0$ as $t\to+0$ for some positive integer p.

By integration by parts

$$\phi_h^*(x) = c_0\phi_h(x)+c_1\phi_{h+1}(x)+\ldots+c_p\phi_{h+p}(x)+ \\ +c_{p+1}x\int\limits_x^\infty \phi_{h+p}(u)u^{h+p}\left(\frac{d}{du}\right)^{h+p}\{(u-x)^{h-1}u^{-h-1}\}\,du. \tag{5.1}$$

Since $\phi_{h+p}(t)\to 0$, it can be shown that

$$x\int\limits_x^\infty \phi_{h+p}(u)u^{h+p}\left(\frac{d}{du}\right)^{h+p}\{(u-x)^{h-1}u^{-h-1}\}\,du\to 0 \tag{5.2}$$

as $x\to+0$. Hence, for $\alpha>h$, the Fourier series of the last two terms of (5.1) are summable $(C,\alpha-h)$ to zero. By Theorem 1, $S[\phi_h^*]$ is summable $(C,\alpha-h)$ to zero, and hence, from (5.1), either

$$S[\phi_h] \quad \text{is summable} \quad (C,\alpha-h) \tag{5.3}$$

or, for some q_1 $(1\leqslant q_1\leqslant h+p-1)$, $S[\phi_{h+q_1}]$ is not summable $(C,\alpha-h)$. Suppose that (5.3) is false. By replacing h in (5.1) by $h+1$, $h+2$,... we can obtain a sequence of integers $1\leqslant q_1<q_2<\ldots$ such that $S[\phi_{h+q_r}]$ is not summable $(C,\alpha-h)$. By the arguments in the necessity part of Theorem 1 this implies that $S[\phi]$ is not summable (C), giving a contradiction. Thus (5.3) is true. Using a similar argument we can show that, if (5.3) holds, then $S[\phi_h^*]$ is summable $(C,\alpha-h)$, and hence by Theorem 1 that $S[\phi]$ is summable (C,α). Thus Theorem A is contained in Theorem 1. The fact that (5.1) and (5.2) cease to hold if h is not an integer may be regarded as a reason for the breakdown of Theorem A for non-integral h.

Finally, if p is a non-negative integer, summability (C,p) and (C^*,p) are equivalent whenever the latter method is applicable. Hence, when α and h are integers, Theorem 2 coincides with Theorem A.

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REFERENCES

1. A. F. Andersen, 'Comparison theorems in the theory of Cesàro summability', *Proc. London Math. Soc.* 27 (1928) 39–71.
2. L. S. Bosanquet, 'On the Cesàro summation of Fourier series and allied series', ibid. (2) 37 (1934) 17–32.
3. —— 'On the summability of Fourier series', ibid. (2) 31 (1930) 144–64.
4. —— 'An extension of a theorem of Andersen', *J. London Math. Soc.* 25 (1950) 72–80.
5. —— 'On Cesàro–Lebesgue integrals', *Proc. London Math. Soc.* (2) 49 (1945) 40–62.
6. L. S. Bosanquet and E. H. Linfoot, 'Note on an asymptotic formula', *Tôhoku Math. J.* 39 (1934) 11–16.
7. J. J. Gergen, 'Summability of double Fourier series', *Duke Math. J.* 3 (1937) 133–48.
8. G. H. Hardy and J. E. Littlewood, 'Solution of the Cesàro summability problem for power series and Fourier series', *Math. Z.* 19 (1924) 67–96.
9. —— 'Notes on the theory of series (III)', *Proc. Cambridge Phil. Soc.* 23 (1927) 681–4.
10. E. W. Hobson, *Theory of functions of a real variable*, vol. 2 (Cambridge, 1926).
11. B. Kuttner, 'On quasi-Hausdorff transformations', *Quart. J. of Math.* (Oxford) (2) 8 (1957) 272–8.
12. E. C. Titchmarsh, *Theory of Fourier integrals* (Oxford 1937).
13. A. J. White, 'On quasi-Cesàro summability', *Quart. J. of Math.* (Oxford) *supra* 81–99.

ANALYSIS OF CUMULATIVE SUMS BY MULTIPLE CONTOUR INTEGRATION

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1. Introduction

THE purpose of this paper is to describe a simple method, using multiple contour integration, for obtaining formulae in the theory of probability of a type due to Pólya, Szegö, Pollaczek, Kac, Hunt, Bohnenblust, Andersen, Baxter, and especially to Spitzer (**1**, **2**, **3**, **4**, **5**, **6**, **7**, **8**, **9**). The formulae are concerned with the partial sums S_n of a sequence of independent and identically distributed random variables. They have several applications, but these will not be discussed. The method resembles that of Pollaczek but is more elementary. A simple method of extending the results to Markov chains is also given.

2. Let the sequence of independent random variables be $\{X_n\}$, so that $S_n = X_1+...+X_n$. We shall suppose that the random variables are integers. This is a severe restriction from the point of view of the pure mathematician, and is not the only restriction we shall make, but even in this limited case the formulae are of interest and our novel and elementary proof may be helpful. The further restriction is stated after equation (1).

Let $P(X_n = m) = p_m$, so that the probability generating function of X_n is a Laurent series

$$f(z) = \sum_{-\infty}^{\infty} p_m z^m,$$

absolutely convergent on the unit circle. Let

$$P(S_m = k) = p_{m,k},$$

so that
$$\sum_{k=-\infty}^{\infty} p_{m,k} z^k = \Big(\sum_{k=-\infty}^{\infty} p_k z^k\Big)^m = \{f(z)\}^m,$$

which we also write as $f^m(z)$ in this paper. Let

$$f^{m-}(z) = \sum_{k=-\infty}^{-1} p_{m,k} z^k, \qquad f^{m+}(z) = \sum_{k=0}^{\infty} p_{m,k} z^k,$$

the 'negative part' and the 'positive part' of the Laurent series for $f^m(z)$. Let

$$p_{n,r,s} = P(S_1 \geqslant -r, S_2 \geqslant -r,..., S_n \geqslant -r, S_n = s-r),$$

the probability, as it were, of arriving at a certain degree of success,

without having been ruined. My main task is to prove the formula [a special case of Spitzer (**6**) 336, Theorem 6.1]:

$$F(x,y,t) = \sum_{r,s,n=0}^{\infty} p_{n,r,s} x^r y^s t^n = \sum_{n=0}^{\infty} F_n(x,y) t^n$$
$$= (1-xy)^{-1} \exp\Big[\sum_{m=1}^{\infty} m^{-1} t^m \{f^{m-}(x^{-1}) + f^{m+}(y)\}\Big]$$
$$(|x| \leqslant 1,\ |y| \leqslant 1,\ |t| < 1) \quad (1)$$

subject to the conventions

$$p_{0,r,r} = 1, \quad p_{0,r,s} = 0 \quad (r \neq s),$$

and on the assumption that $f(z)$ is regular in an open annulus containing the unit circle. The part in braces { } may also be written

$$g_m(x^{-1}) + h_m(y) - 1,$$

where $g_m(y)$ and $h_m(y)$ are the probability generating functions of $\min(0, S_m)$ and $\max(0, S_m)$.

Interesting special cases (seventeen in all) are obtained by putting $y = 0$ or 1, $x = 0$ or 1, and $t = 1$. For example,

$$\sum_{s,n \geqslant 0} P(S_1 \geqslant 0, \ldots, S_n \geqslant 0, S_n = s) y^s t^n = \exp\Big\{\sum_1^{\infty} m^{-1} t^m f^{m+}(y)\Big\}, \quad (2)$$

$$\sum_{n=0}^{\infty} P(S_1 \geqslant 0, \ldots, S_n \geqslant 0) t^n = \exp\Big\{\sum_1^{\infty} m^{-1} t^m P(S_m \geqslant 0)\Big\}, \quad (3)$$

(a formula due to Pollaczek and Andersen),

$$\sum_{r,n \geqslant 0} P(S_1 \geqslant -r, \ldots, S_n \geqslant -r) x^r t^n = (1-x)^{-1} \exp\Big\{\sum_1^{\infty} m^{-1} t^m g_m(x^{-1})\Big\}, \quad (4)$$

and

$$\sum_{n=0}^{\infty} P(S_1 \geqslant 0, \ldots, S_{n-1} \geqslant 0, S_n = 0) t^n = \exp\Big\{\sum_1^{\infty} m^{-1} t^m P(S_m = 0)\Big\}. \quad (5)$$

Equation (5) gives the generating function for 'recurrent right-handed walks', in the terminology of Good (**10**, **11**), where the special case of the 'trinomial random walk' ($p_{-1}+p_0+p_1 = 1$) was considered in detail and can be used as a check. The identity reduces to one involving Legendre polynomials.

The effect of putting $t = 1$ in some of these formulae is to obtain formulae for the expected number of various kinds of 'success' in an infinite sequence of trials. These expectations are sometimes infinite.

All numbered formulae in this paper apply practically without change to Markov chains, for the reason given in § 5. The only previous work known to me, in this general area, concerned with Markov chains, is

due to Baxter (9). His methods are abstract and completely different from mine, and his results also seem to me to be different, as far as I can understand them.

3. Proof of formula (1)

Consider the multiple contour integral

$$\left(\frac{1}{2\pi i}\right)^n \int \cdots \int \frac{f(z_1)\ldots f(z_n)\,dz_1\,dz_2\ldots dz_n}{(1-xz_1)(z_1-x_1 z_2)\ldots(z_{n-1}-x_{n-1} z_n)(z_n-x_n)},$$

where the contours will be assumed, *both here and later*, to be circular, with centres at the origins. Since we have assumed $f(z)$ to be regular in an open annulus containing the unit circle, we may take $|z_1|$, $|z_2|,\ldots,$ $|z_n|$ all close to 1 on the contours, and with

$$|z_1| > |z_2| > \ldots > |z_n|.$$

Let the x's be such that, when the z's are on these circles, we have

$$|x| < |z_1|^{-1}, \quad |x_1| < |z_1|/|z_2|, \quad \ldots, \quad |x_{n-1}| < |z_{n-1}|/|z_n|, \quad |x_n| < |z_n|.$$

Then we may expand the integrand in ascending powers of the x's. The coefficient of

$$x^r x_1^{s_1} \ldots x_n^{s_n} \quad (r \geqslant 0,\ s_1 \geqslant 0,\ldots,\ s_n \geqslant 0)$$

in the integral itself is equal to

$$\begin{aligned} p_{s_1-r}\, p_{s_2-s_1} \cdots p_{s_n-s_{n-1}} &= P(X_1 = s_1-r,\ X_2 = s_2-s_1,\ldots,\ X_n = s_n-s_{n-1}) \\ &= P(S_1 = s_1-r,\ S_2 = s_2-r,\ldots,\ S_n = s_n-r). \end{aligned}$$

Now put $x_1 = \ldots = x_{n-1} = 1$, a substitution which is consistent with the above inequalities, and we see that

$$F_n = F_n(x,y) = \left(\frac{1}{2\pi i}\right)^n \int \cdots \int \frac{f(z_1)\ldots f(z_n)\,dz_1\ldots dz_n}{(1-xz_1)(z_1-z_2)\ldots(z_{n-1}-z_n)(z_n-y)}, \tag{6}$$

where $\qquad |x|^{-1} > |z_1| > \ldots > |z_n| > |y|$

on the contours.

Now, if $g(z)$ is a Laurent series around the origin, we have

$$\frac{1}{2\pi i} \int \frac{g(z)\,dz}{a-z} = \begin{cases} g^-(a) & (|a| > |z|), \\ -g^+(a) & (|z| > |a|) \end{cases} \tag{7}$$

when the specified inequalities apply over the whole contour. Therefore we may see by a partial-fraction decomposition that

$$\frac{1}{2\pi i} \int \frac{g(z)\,dz}{(a-z)(z-b)} = \frac{g^-(a)+g^+(b)}{a-b} \quad (|a| > |z| > |b|). \tag{8}$$

Let us apply (8) to the contour integral

$$F_{n,m,r} = F_{n,m,r}(x,y)$$
$$= \left(\frac{1}{2\pi i}\right)^n \int \dots \int \frac{f(z_1)\dots f(z_{r-1})f^m(z_r)f(z_{r+1})\dots f(z_n)\,dz_1\dots dz_n}{(1-xz_1(z_1-z_2)\dots(z_{n-1}-z_n)(z_n-y)},$$

with $z = z_r$, $a = z_{r-1}$, $b = z_{r+1}$ $(1 < r < n)$; and afterwards change the notation from $z_{r+1},\dots, z_n$ to $z_r,\dots, z_{n-1}$. We get

$$F_{n,m,r} = \left(\frac{1}{2\pi i}\right)^{n-1} \int \dots \int \frac{f(z_1)\dots f(z_{n-1})}{(1-xz_1)\dots(z_{n-1}-y)} \times$$
$$\times \{f^{m-}(z_{r-1})+f^{m+}(z_r)\}\,dz_1\dots dz_{n-1}.$$

When $r = 1$ and $r = n$, the factors in the integrand in braces $\{\}$ are respectively

$$f^{m-}(x^{-1})+f^{m+}(z_1) \quad \text{and} \quad f^{m-}(z_{n-1})+f^{m+}(y).$$

The sum of the contents of the n braces is

$$f^{m-}(x^{-1})+f^{m}(z_1)+\dots+f^{m}(z_{n-1})+f^{m+}(y).$$

Hence
$$\sum_{r=1}^{n} F_{n,m,r} = \gamma_m F_{n-1} + \sum_{r=1}^{n-1} F_{n-1,m+1,r} \quad (n \geqslant 2), \tag{9}$$

where
$$\gamma_m = f^{m-}(x^{-1})+f^{m+}(y).$$

(Note that $|\gamma_m| \leqslant 1$ when $|x| \leqslant 1$, $|y| \leqslant 1$ since $f^m(z)$ is a probability generating function.) Now, by (8),

$$F_{1,m,1} = \gamma_m/(1-xy),$$

and so (9) is true also when $n = 1$ if we define F_0 as $1/(1-xy)$. Therefore, by induction with respect to n,

$$\sum_{r=1}^{n} F_{n,m,r} = \gamma_m F_{n-1} + \gamma_{m+1} F_{n-2} + \dots + \gamma_{m+n-1} F_0, \tag{10}$$

and, in particular ($m = 1$),

$$nF_n = \gamma_1 F_{n-1} + \gamma_2 F_{n-2} + \dots + \gamma_n F_0. \tag{11}$$

But, if
$$\exp\left(\sum_{m=1}^{\infty} m^{-1} t^m \gamma_m\right) = \sum_{0}^{\infty} B_n t^n,$$

we know, by differentiation, that the B_n satisfy the linear recurrence relation (11), with $B_0 = 1$. (The series are absolutely convergent when $|t| < 1$, since $|\gamma_m| \leqslant 1$.) Hence

$$\sum_{n=0}^{\infty} (1-xy)F_n t^n = \exp\left(\sum_{1}^{\infty} \frac{t^m \gamma_m}{m}\right),$$

which is the same thing as equation (1).

4. Miscellaneous extensions

A generalization of formula (6), which can be proved by the method, is

$$\left(\frac{1}{2\pi i}\right)^n \int \cdots \int \frac{f(z_1, z_2,..., z_n)\, dz_1\, dz_2 \dots dz_n}{\prod\limits_{\nu=0}^{n} \{z_\nu^{k_\nu-1}(z_{\nu+1}-\xi_\nu z_\nu)(z_\nu-\eta_\nu z_{\nu+1})\}}$$

$$= \sum_{r_0,...,s_n=0}^{\infty} \{P(S_1 = r_0-s_0+s_1-r_1+k_1,\ S_2 = r_0-s_0+s_2-r_2+k_1+k_2,...,$$
$$S_n = r_0-s_0+s_n-r_n+k_1+...+k_n)\xi_0^{r_0} \dots \xi_n^{r_n}\eta_0^{s_0} \dots \eta_n^{s_n}\}, \quad (12)$$

where k_0, k_1,..., k_n are integers, X_0,..., X_n are integer-valued random variables with joint probability generating function $f(z_1, z_2,..., z_n)$ which is regular in an open domain containing the circular polycylinder $|z_1| = ... = |z_n| = 1$, z_0 and z_{n+1} are new notations for the number 1, the contours are (as always in this paper) circles with centres at the origins and of radii close to 1, and on these contours

$$|\xi_\nu| < |z_{\nu+1}/z_\nu|, \qquad |\eta_\nu| < |z_\nu/z_{\nu+1}| \quad (\nu = 0, 1,..., n).$$

Example (i). Assume that $|z_1| > |z_2| > ... > |z_n|$ on the contours and take

$$k_1 = k_2 = ... = k_n = q, \qquad \xi_0 = \xi_1 = ... = \xi_n = 0,$$
$$\eta_0 = x, \qquad \eta_1 = \eta_2 = ... = \eta_{n-1} = 1, \qquad \eta_n = y,$$
$$f(z_1, z_2,..., z_n) = f(z_1)...f(z_n), \qquad |x| < |z_1|^{-1}, \qquad |y| < |z_n|$$

on the contours, and let us write $s_0 = r$, $s_n = s$. Then we get

$$\left(\frac{1}{2\pi i}\right)^n \int \cdots \int \frac{f(z_1)...f(z_n)\, dz_1 \dots dz_n}{(z_1 \dots z_n)^q(1-xz_1)(z_1-z_2)\dots(z_{n-1}-z_n)(z_n-y)}$$
$$= \sum_{r,s=0}^{\infty} P(S_1 \geqslant -r+q,\ S_2 \geqslant -r+2q,...,\ S_n \geqslant -r+nq,$$
$$S_n = s-r+nq)x^r y^s.$$

But the integrand is obtainable from (6) by replacing $f(z)$ by $f(z)z^{-q}$. So, by an obvious extension of the proof of equation (1), as given in § 3, we have

$$\sum_{r,s,n=0}^{\infty} P(S_1 \geqslant -r+q, S_2 \geqslant -r+2q,..., S_n \geqslant -r+nq, S_n = s-r+nq$$
$$= (1-xy)^{-1}\exp\Big[\sum_{m=1}^{\infty} m^{-1}t^m\{x^{mq}f_{mq}^{m-}(x^{-1})+y^{-mq}f_{mq}^{m+}(y)\}\Big]$$
$$(|x| \leqslant 1,\ |y| \leqslant 1,\ |t| < 1), \quad (13)$$

where $f_{mq}^{m+}(y)$, for example, represents the Laurent series for $f^m(y)$ truncated at the power y^{mq}, i.e. with the terms to the left of this power omitted (mq is here a product, not a pair of suffixes).

Example (ii). In equation (12) take

$$n = 5,\quad \xi_0 = 0,\quad \eta_0 = 0,\quad \xi_1 = 1,\quad \eta_1 = 0,\quad \xi_2 = 0,$$
$$\eta_2 = 1,\quad \xi_3 = 0,\quad \eta_3 = 1,\quad \xi_4 = 1,\quad \eta_4 = 0,\quad \xi_5 = 0,\quad \eta_5 = y,$$
$$s_5 = s,\quad k_1 = -1,\quad k_2 = 1,\quad k_3 = 0,\quad k_4 = -1,\quad k_5 = 1,$$

and we see that

$$\left(\frac{1}{2\pi i}\right)^5 \int \cdots \int \frac{f(z_1,\ldots,z_5)\,dz_1\ldots dz_5}{(z_2-z_1)(z_2-z_3)(z_3-z_4)(z_5-z_4)(z_5-y)}$$
$$= \sum_{s=0}^{\infty} P(S_1 < 0, S_2 \geqslant 0, S_3 \geqslant 0, S_4 < 0, S_5 = s)y^s \quad (14)$$

if $|z_2| > |z_1|$, $|z_2| > |z_3|$, $|z_3| > |z_4|$, $|z_5| > |z_4|$, $|z_5| > |y|$ on the contours, and the contours have radii close to 1.

Example (iii).

$$P(S_1 < 0, S_4 \geqslant 0, S_5 \geqslant 0)$$
$$= \left(\frac{1}{2\pi i}\right)^3 \int\int\int \frac{f(z_1, z_2, z_2, z_2, z_5)\,dz_1\,dz_2\,dz_5}{(z_2-z_1)(z_2-z_5)(z_5-1)}, \quad (15)$$

where $|z_2| > |z_1|$, $|z_2| > |z_5| > 1$ on the contours, the contours have radii close to 1, and f is regular in an open domain containing

$$|z_1| = |z_2| = |z_5|.$$

Integrals like (14) and (15) can be evaluated by repeated integration, in the order $z_n, z_{n-1},\ldots, z_1$, by making use of (8) and of the corresponding results for the cases $|z| > |a|$, $|z| > |b|$, and $|a| > |z|$, $|b| > |z|$. When the variables $X_1, X_2,\ldots, X_n$ are independent, this method would lead to the same calculations as the straightforward elementary numerical evaluation of the left-hand sides. A method of approximating to the multiple integrals is to replace each contour by a regular polygon and sum over the vertices, or to go back in some other way to the definition of a Riemann integral. In the case of independence, each summation involves only one parameter: for example, z_{n-1} is the parameter when summing with respect to z_n. Sometimes this method would involve less and sometimes more arithmetic than the straightforward method.

A further formula which expresses the probability that the cumulative sums will exceed assigned thresholds in an n-stage sequential procedure is

$$P(S_1 \geqslant t_1,\ldots, S_n \geqslant t_n)$$
$$= \left(\frac{1}{2\pi i}\right)^n \int \cdots \int \frac{f_1(z_1)\ldots f_n(z_n)\,dz_1\ldots dz_n}{z_1^{t_1} z_2^{t_2-t_1}\ldots z_n^{t_n-t_1}(z_1-z_2)\ldots(z_{n-1}-z_n)(z_n-1)}, \quad (16)$$

where $t_1, t_2, \ldots, t_n$ are integers, $|z_1| > |z_2| > \ldots > |z_n| > 1$ on the contours, and each $f_\nu(z)$ is regular in an annulus containing $|z| = 1$. This assumes that $X_1, X_2, \ldots, X_n$ are independent but not necessarily identically distributed. If they are not independent, then the product of the f's is to be replaced by the joint probability generating function.

5. Markov chains

Let $q_{\mu,\nu}$ be the transition probability from μ to ν ($\mu, \nu = \ldots, -1, 0, 1, \ldots$) in a stationary Markov chain, so that

$$q_{\mu,\nu} = P(X_{m+1} = \nu | X_m = \mu) \quad (m = 0, 1, 2, \ldots);$$

and define the matrix

$$\mathbf{Q}(z) = \{q_{\mu\nu} z^\nu\} \quad (\mu, \nu = \ldots, -1, 0, 1, \ldots).$$

The nth power of $\mathbf{Q}(z)$ is readily seen to be equal to

$$\mathbf{Q}^n(z) = \left\{ \sum_{s=-\infty}^{\infty} P(S_n = s, X_n = \nu \mid X_0 = \mu) z^s \right\}. \tag{17}$$

$\mathbf{Q}^n(1)$ is the usual n-stage transition probability matrix in which the information concerning the partial sums is suppressed. It is natural to think of $\mathbf{Q}(z)$ as a *probability generating matrix.*

Let $\mathbf{Q}^+(z)$ be obtained from $\mathbf{Q}(z)$ by suppressing all the negative powers of z in each term of the matrix; let $\mathbf{Q}^-(z) = \mathbf{Q}(z) - \mathbf{Q}^+(z)$, and let $\mathbf{Q}^{n+}(z)$ and $\mathbf{Q}^{n-}(z)$ be similarly defined.

In the previous discussion we must replace $p_{n,r,s}$ by the matrix

$$\{P(S_1 \geqslant r, \ldots, S_n = s-r, X_n = \nu \mid X_0 = \mu)\}.$$

Then all of the discussion on independent random variables in the previous sections goes through almost unchanged, apart from the replacement of various scalars by matrices. In the integrands the matrices appear only in the numerators. Note that ν can be summed out in all numbered formulae (1) to (12) (after they have been modified); and that the most appropriate value of μ is zero in order that there should be no ambiguity concerning the expression 'partial sum'. Thus scalar equations can be obtained, almost identical with the original ones. For an example, see the last sentence in the next section.

6. Generalized Markov chains

Suppose that at the rth stage the probability of going from state μ to state ν is $p_{\mu\nu}^{(r)}$. (For an ordinary Markov chain this probability does not depend on r.) Write

$$\mathbf{Q}_r(z) = \{q_{\mu\nu}^{(r)} z^\nu\},$$

the probability generating matrix at the rth stage. Then

$$\mathbf{Q}_1(z)\mathbf{Q}_2(z)\ldots\mathbf{Q}_n(z) = \left\{\sum_{s=-\infty}^{\infty} P(S_n = s, X_n = \nu \mid X_0 = \mu)z^s\right\}, \quad (18)$$

which generalizes equation (17). Thus the discussion of § 5 goes through with $\mathbf{Q}^n$ replaced by the product of the $\mathbf{Q}_r$'s, but there will now be no obvious analogue of equation (1). There will be a natural analogue of equation (12) and of its special cases. For example, the analogue of equation (16) is

$$\{P(S_1 \geqslant t_1, \ldots, S_n \geqslant t_n, X_n = \nu \mid X_0 = \mu)\}$$
$$= \left(\frac{1}{2\pi i}\right)^n \int \cdots \int \frac{\mathbf{Q}_1(z_1)\ldots\mathbf{Q}_n(z_n)\,dz_1\ldots dz_n}{z_1^{t_1}z_2^{t_2-t_1}\ldots z_n^{t_n-t_1}(z_1-z_2)\ldots(z_{n-1}-z_n)(z_n-1)}. \quad (19)$$

The corresponding result for an ordinary Markov chain may be written down as a special case.

REFERENCES

1. F. Pollaczek, 'Fonctions caractéristiques de certains répartitions définies au moyen de la notion d'ordre. Application à la théorie des attentes', *C.R. Acad. Sci. Paris*, 234 (1952) 2334–6; review in *Math. Rev.* 13. 957.
2. —— 'Sur la répartition des périodes d'occupation ininterrompue d'un guichet', *C.R. Acad. Sci. Paris*, 234 (1952) 2042–4; *MR* 14. 188.
3. E. S. Andersen, 'On the fluctuations of sums of random variables', *Math. Scand.* 1 (1953) 263–85; *MR* 15. 444; *Math. Scand.* 2 (1954) 195–223; *MR* 16. 839; *Math. Scand.* 2 (1954) 193–4; *MR* 16. 839.
4. M. Kac, 'Toeplitz matrices, translation kernels and a related problem in probability theory', *Duke Math. J.* 21 (1954) 501–9; *MR* 16. 31.
5. —— *Probability and related topics in physical sciences* (London–New York, 1959); *MR* 21. 314.
6. F. Spitzer, 'A combinatorial lemma and its application to probability theory', *Trans. American Math. Soc.* 82 (1956) 323–39; *MR* 18. 156.
7. —— 'The Wiener–Hopf equation whose kernel is a probability density', *Duke Math. J.* 24 (1957) 327–43; *MR* 19. 890.
8. J. G. Wendel, 'Spitzer's formula: a short proof', *Proc. American Math. Soc.* 9 (1958) 905–8; *MR* 21. 441.
9. G. Baxter, 'An operator identity', *Pacific J. Math.* 8 (1958) 649–63; *MR* 21. 441.
10. I. J. Good, 'Legendre polynomials and trinomial random walks', *Proc. Cambridge Phil. Soc.* 54 (1958) 39–42; *MR* 21. 442.
11. —— 'Random motion and analytic continued fractions', *Proc. Cambridge Phil. Soc.* 54 (1958) 43–47; *MR* 21. 442.

ASYMPTOTIC STABILITY OF $x''+a(t)x'+x=0$

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1. This paper concerns the differential equation

$$x''+a(t)x'+x=0, \tag{1}$$

where $a(t)\geqslant 0$ is continuous in $\theta\leqslant t<\infty$. The equation is *asymptotically stable* if every solution $x(t)$ has $x, x'\to 0$ as $t\to+\infty$. The following results are proved:

Theorem 1. *If* (1) *is asymptotically stable, then*

$$\infty=\int_\theta^\infty a(t)\,dt \tag{2}$$

and

$$\infty=\int_\theta^\infty e^{-A(t)}\int_\theta^t e^{A(s)}\,dsdt, \tag{3}$$

where

$$A(t)=\int_\theta^t a(s)\,ds. \tag{4}$$

Theorem 2. *Equation* (1) *is asymptotically stable if* (3) *holds and there exist constants* $\epsilon>0$ *and* $t_0\geqslant\theta$ *such that* $a(t)\geqslant\epsilon$ *for all* $t\geqslant t_0$.

Theorem 3. *Equation* (1) *is asymptotically stable if there exists a sequence* $I_1, I_2,\ldots$ *of disjoint open intervals in* (θ,∞) *such that*

$$\infty=\sum_{n=1}^\infty m_n T_n \delta_n^2, \tag{5}$$

where m_n *and* M_n *are the minimum and maximum values of* $a(t)$ *in* I_n, T_n *is the length of* I_n *and* δ_n *is the smaller of the two numbers* T_n *and* $(1+M_n)^{-1}$.

Corollary. *Equation* (1) *is asymptotically stable if* $a(t)$ *is monotonic decreasing in* (θ,∞) *and* (2) *holds.*

Condition (3) of Theorem 1 follows immediately from a theorem of Wintner's (1). As suggested by Bellman (2), certain equations of the type (1) can be studied by transforming them into the adiabatic oscillator equation, about which a good deal is known. This produces results of restricted generality however and none of the above appears to be obtainable in this way. Theorem 3 resembles a theorem of Sansone's

(**3**) concerning the adiabatic oscillator equation, but the present writer cannot deduce either of these results from the other.

2. *Proof of Theorem* 1. If $E = x^2+(x')^2$, then (1) gives

$$\frac{dE}{dt} = -2a(x')^2, \tag{6}$$

$$\log\frac{E(T)}{E(\theta)} = -\int_{\theta}^{T} \frac{2a(x')^2}{E}\,dt \geqslant -2\int_{\theta}^{T} a\,dt.$$

If $E(T) \to 0$ as $T \to \infty$, then (2) follows from this.

Wintner's theorem (**1**) states that, if $p(t)$, $q(t)$ are continuous, $p(t) \neq 0$, $q(t) \not\equiv 0$ in (θ,∞), and

$$\infty > \int_{\theta}^{\infty}\int_{\theta}^{t} |q(s)|ds/\,|p(t)|\,dt, \tag{7}$$

then every solution $x(t)$ of the differential equation

$$(p(t)x')'+q(t)x = 0 \tag{8}$$

tends to a finite limit $x(\infty)$ as $t \to +\infty$ and $x(\infty) \neq 0$ for at least one solution. If $p = q = e^{A}$, then (8) is equivalent to (1) and, if this is asymptotically stable, (7) is false and (3) holds.

Proof of Theorem 2. It is implicit in Wintner's work that, if $p > 0$, $q > 0$ in (θ,∞), then (7) is also a *necessary* condition for (8) to have a solution with $x(\infty) \neq 0$. The conditions $p > 0$, $q > 0$ are satisfied when $p = q = e^{A}$ and, if (3) holds, (1) cannot have a solution with $x(\infty) \neq 0$. If Theorem 2 is false, (1) has a solution $x(t)$ for which $E(t) \not\to 0$ as $t \to \infty$. By (6), $E(t)$ is monotonic decreasing and tends to a limit $\eta^2 > 0$. Notice that, if $x' \to 0$, then $x \to \pm\eta$ as $t \to \infty$, which is impossible by Wintner's work since (3) holds. Hence

$$0 < \Lambda = \limsup|x'|.$$

The values of $t > t_0$ for which $|x'(t)| > \frac{1}{3}\Lambda$ form an unbounded set Ω which is open because $x'(t)$ is continuous. By integrating (6) we obtain

$$E(t_0) \geqslant \int_{t_0}^{\infty} 2a(x')^2\,dt > \tfrac{2}{3}\Lambda\int_{\Omega} a|x'|\,dt > \tfrac{2}{9}\Lambda^2\epsilon\int_{\Omega} dt. \tag{9}$$

This shows in particular that Ω has finite measure. Since it is unbounded, it must be the union of an infinite sequence $\{\omega_n\}$ of disjoint

open intervals and (9) shows that

$$\int_{\omega_n} a|x'|\,dt, \qquad \int_{\omega_n} dt$$

both tend to zero as $n \to \infty$. For an infinite number of the intervals $\{\omega_n\}$, $|x'|$ must rise from the value $\frac{1}{3}\Lambda$ at the initial point α_n to the value $\frac{2}{3}\Lambda$ at some interior point β_n. Since $|x|$ is bounded above by $K = E^{\frac{1}{2}}(t_0)$, (1) gives

$$\tfrac{1}{3}\Lambda = \left|\int_{\alpha_n}^{\beta_n} x''\,dt\right| \leqslant \int_{\omega_n} a|x'|\,dt + K\int_{\omega_n} dt,$$

for an infinite number of values of n. Then $\Lambda = 0$ because the right-hand side tends to zero as $n \to \infty$. This contradicts $\Lambda > 0$ and proves Theorem 2.

Proof of Theorem 3. If Theorem 3 is false, (1) has a solution $x(t)$ for which $E(t)$ tends to a non-zero limit as $t \to \infty$ and by scaling it suitably we can suppose that $2 < E < K^2$ (K constant) for $\theta \leqslant t < \infty$. This and (1) give

$$K + M_n|x'| \geqslant |x''| \geqslant 1 - M_n|x'| \tag{10}$$

when $|x'| \leqslant 1$ and $a(t) \leqslant M_n$. We first show that $|x'| > \mu_n$ in at least half of the interval I_n, where

$$\mu_n = \min\{(1+M_n)^{-1}(4K+5)^{-1}, \tfrac{1}{8}T_n\}. \tag{11}$$

In any sub-interval U of I_n in which $|x'| \leqslant \mu_n$, (10) and (11) give $|x''| > \frac{1}{2}$ (crudely), which shows that $x'(t)$ is monotonic in U and that the length of U does not exceed $4\mu_n$. This is less than $\frac{1}{2}T_n$ by (11). Either $|x'| > \mu_n$ throughout the remainder of I_n, or I_n contains several U-intervals. Between any two of these U-intervals there is at least one interval J in which $|x'| > \mu_n$. Throughout any part of J in which

$$\mu_n < |x'| < (1+M_n)^{-1},$$

(10) shows that

$$0 < |x''| < K+1,$$

so that $x'(t)$ is monotonic in that part of J and its length exceeds

$$[(1+M_n)^{-1} - \mu_n]/(K+1).$$

There is such a part at the beginning of J in which $|x'|$ increases from μ_n to $(1+M_n)^{-1}$ and another such part at its end in which $|x'|$ decreases from $(1+M_n)^{-1}$ to μ_n. The length of J therefore exceeds

$$2\{(1+M_n)^{-1} - \mu_n\}/(K+1),$$

and this exceeds $8\mu_n$ by (11). Thus I_n contains only a finite number of J-intervals and therefore a finite number of U-intervals also. The sum

of the lengths of a consecutive pair of U-intervals does not exceed $8\mu_n$, which is less than the length of the J-interval separating them. This proves that $|x'| > \mu_n$ for at least half of I_n.

By (6) the decrease in E during the interval I_n is

$$\Delta_n E = \int_{I_n} 2a(x')^2\,dt \geqslant 2m_n \int_{I_n} (x')^2\,dt > m_n T_n \mu_n^2.$$

Then $$\sum^{\infty} \Delta_n E, \qquad \sum^{\infty} m_n T_n \mu_n^2$$

are both convergent because E is positive and decreasing for all $t \geqslant \theta$. Since $\mu_n \geqslant (4K+8)^{-1}\delta_n$ by (11), $\sum^{\infty} m_n T_n \delta_n^2$ is also convergent. This contradicts (5) and establishes Theorem 3.

Proof of Corollary. In Theorem 3 take I_n to be the interval $(\theta+n-1, \theta+n)$. Then

$$T_n = 1, \quad m_n = a(\theta+n), \quad M_n < a(\theta), \quad \delta_n > [1+a(\theta)]^{-1}$$

since $a(t)$ is monotonic decreasing. For (5) it is sufficient to have $\sum^{\infty} a(\theta+n)$ divergent. This is equivalent to (2) because $a(t)$ is monotonic.

3. I am indebted to Professor G. E. H. Reuter for suggesting the problem and for helpful discussions.

REFERENCES

1. A. Wintner, 'Asymptotic integrations of the adiabatic oscillator in its hyperbolic range', *Duke Math. J.* 15 (1948) 55–67.
2. R. Bellman, *Stability theory of differential equations* (New York, 1953) 137.
3. G. Sansone, *Equazioni differenziali nel campo reale: parte seconda* (2nd ed. Bologna, 1949) 66.

ON REGULAR AUTOMORPHISMS OF CERTAIN CLASSES OF RINGS

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1. Introduction

An automorphism T of a ring $\mathbf{R}$ (not necessarily associative) is said to be *regular* if it leaves fixed no non-zero element of $\mathbf{R}$. Graham Higman has shown that, if T is of prime order and $\mathbf{R}$ is an associative ring or a Lie ring, then $\mathbf{R}$ is nilpotent [(**4**); see also (**5**)]. The present paper arises from a consideration of how these results are modified when T is not of prime order. Some of the theorems are concerned with cases in which T is of finite degree; certain properties of such automorphisms have already been discussed elsewhere (**8**).

In the case of associative rings of prime or zero characteristic, we prove by means of an elementary argument that the existence of a regular automorphism of finite degree implies nilpotency. For Lie rings, Higman's theorem does not extend in the same way. As Borel and Mostow have shown (**3**), a finite-dimensional Lie algebra admitting a regular automorphism need not be nilpotent. However, one of our results (Theorem 5) is that a Lie algebra of finite dimension over a field of characteristic zero is soluble if it admits a regular automorphism. It is possible to deduce this theorem from some of the results proved by Borel and Mostow,† but we shall appeal to their paper only for the information that a finite-dimensional simple Lie algebra over a field of characteristic zero cannot admit a regular automorphism. The result then follows from the main theorem (Theorem 3). This states effectively that, in classes of rings which are closed under the operation of taking homomorphic images and have transformation rings satisfying the minimum condition, the problem of deciding whether there exists a ring which admits a certain type of automorphism and does not coincide with its own radical can be reduced to the problem of deciding whether there exists a simple ring admitting a regular automorphism. As well

† Borel and Mostow make use of semi-simple automorphisms. If an algebra (not necessarily finite-dimensional) admits a regular automorphism of finite degree, then it admits a semi-simple regular automorphism: that is, one whose minimal equation has no repeated factors. This is very easily proved, but in this paper no direct use is made of this property.

as applying this theorem to Lie algebras we use it to obtain results concerning associative rings satisfying the minimum condition (Theorem 4) and Jordan algebras of finite dimension (Theorem 6). In both these cases we prove the nilpotency of the rings under consideration; solubility rather than nilpotency arises in the Lie case because the radical of a Lie algebra is soluble but not necessarily nilpotent.

In the final section we prove that the above results no longer hold, even in the associative case, when finiteness restrictions are completely removed. In fact we prove that, if a variety of rings contains simple rings, then it contains semi-simple rings admitting regular automorphisms.

Throughout the present paper the word 'ring' without qualification is taken to mean a non-associative ring, in the sense of being not necessarily associative. A *simple* ring $\mathbf{R}$ is one which contains no proper non-zero (two-sided) ideals and is such that $\mathbf{R}^2 \neq \mathbf{0}$. A *semi-simple* ring is one which is a direct sum of simple rings. If $\mathbf{R}$ contains an ideal $\mathbf{N}$ such that $\mathbf{R}-\mathbf{N}$ is semi-simple or zero and $\mathbf{N}$ is contained in any ideal $\mathbf{S}$ such that $\mathbf{R}-\mathbf{S}$ is semi-simple, then, for the purposes of the present paper, $\mathbf{N}$ is called the *radical* of $\mathbf{R}$. If $\mathbf{R}$ is a ring whose transformation ring satisfies the minimum condition, then the radical of $\mathbf{R}$ exists [(**7**); see also (**1**)]. For associative rings satisfying the minimum condition, the radical defined in the above manner coincides with that defined in the standard way (**6**).

2. Associative algebras admitting regular automorphisms of finite degree

THEOREM 1. *Let* $\mathbf{R}$ *be an associative ring of prime or zero characteristic, admitting a regular automorphism of finite degree. Then* $\mathbf{R}$ *is nilpotent.*

Proof. There is no loss of generality in assuming that $\mathbf{R}$ is a linear algebra over an algebraically closed field F since any ring of prime or zero characteristic can be embedded in such an algebra and a regular automorphism of the ring extended to a regular automorphism of the algebra. We therefore consider an algebra $\mathbf{A}$ over F with an automorphism $T\colon \mathbf{A} \to \mathbf{A}$ having a minimal equation of the form

$$(T-\lambda_1)^{r_1}\dots(T-\lambda_k)^{r_k} = 0, \tag{1}$$

where $\lambda_1,\dots, \lambda_k$ are distinct non-zero elements of F and $r_1,\dots, r_k$ are positive integers. Let $\mathbf{A}(\lambda_i)$ be the subspace of elements $x \in \mathbf{A}$ such that $(T-\lambda_i)^{r_i}x = 0$. Then the space $\mathbf{A}$ can be expressed as the direct sum of the subspaces $\mathbf{A}(\lambda_i)$ ($i = 1,\dots, k$). Since T is an automorphism of $\mathbf{A}$,

we have [(3), (8)]
$$\mathbf{A}(\lambda_i)\mathbf{A}(\lambda_j) \subseteq \mathbf{A}(\lambda_i\lambda_j),$$
where $\mathbf{A}(\lambda_i\lambda_j)$ is defined to be the zero subspace if $\lambda_i\lambda_j$ is not one of $\lambda_1,..., \lambda_k$.

Suppose that there is a non-zero product
$$x_1 x_2 \dots x_{k+1}, \tag{2}$$
where $x_i \in \mathbf{A}(\lambda_{\alpha_i})$. Then no product of the form
$$x_1 x_2 \dots x_r \quad (r \leqslant k+1)$$
is zero, and so each product
$$\lambda_{\alpha_1}\lambda_{\alpha_2}\dots\lambda_{\alpha_r} \quad (r = 1,..., k+1)$$
is in the set $\lambda_1,..., \lambda_k$. Since there are $k+1$ such products, at least two of them are equal; hence
$$\lambda_{\alpha_{p+1}}\dots\lambda_{\alpha_q} = 1$$
for some integers p, q such that
$$1 \leqslant p < q \leqslant k+1.$$
Since T is regular, none of $\lambda_1,..., \lambda_k$ is equal to unity. Therefore
$$x_{p+1}\dots x_q = 0,$$
which contradicts the hypothesis that the product (2) is non-zero. It follows that every product of $k+1$ elements of the spaces $\mathbf{A}(\lambda)$ is zero, and so, since $\mathbf{A}$ is the direct sum of these spaces, every product of $k+1$ elements of $\mathbf{A}$ is zero. Therefore $\mathbf{A}$ is nilpotent, of class at most k.

3. Automorphisms induced by regular automorphisms

Let $\mathbf{R}$ be a ring and $\mathbf{S}$ an ideal of $\mathbf{R}$. If T is an automorphism of $\mathbf{R}$ such that $T(\mathbf{S}) = \mathbf{S}$, then there is an induced automorphism T^* of $\mathbf{R}-\mathbf{S}$ defined by
$$T^*(x+\mathbf{S}) = T(x)+\mathbf{S}. \tag{3}$$
This automorphism is not necessarily regular when T is regular. In the following theorem, however, we consider a case in which the property of being regular does extend from T to T^*.

THEOREM 2. *Let* $\mathbf{R}$ *be a ring and* $\mathbf{S}$ *an ideal of* $\mathbf{R}$. *If* T *is a regular automorphism of* $\mathbf{R}$ *such that* $T(\mathbf{S}) = \mathbf{S}$ *and* $(T-E)\mathbf{S} = \mathbf{S}$, *where* $E: \mathbf{R} \to \mathbf{R}$ *is the identity, then the automorphism* $T^*: \mathbf{R}-\mathbf{S} \to \mathbf{R}-\mathbf{S}$ *induced by* T *is regular.*

Proof. Suppose that T^* is not regular. Then there exists an element x such that
$$T(x)+\mathbf{S} = x+\mathbf{S} \quad (x \in \mathbf{R};\ x \notin \mathbf{S}).$$
We have $(T-E)x \in \mathbf{S}$; but $\mathbf{S} = (T-E)\mathbf{S}$, and so there exists an

element y of S such that $(T-E)x = (T-E)y$. Therefore T leaves $x-y$ fixed. Since T is regular, it follows that $x = y$, which contradicts $x \notin \mathbf{S}$ and $y \in \mathbf{S}$. Thus Theorem 2 is proved.

Carrying the argument used in this proof a stage further, we see that, if T is a regular automorphism of $\mathbf{R}$ such that $T(\mathbf{S}) = \mathbf{S}$ and the induced automorphism T^* is not regular, then $\mathbf{S}$ contains a descending chain of additive subgroups $\mathbf{S}_i = (T-E)^i\mathbf{S}$, each invariant under T and such that $\mathbf{S}_{i+1}$ is properly contained in $\mathbf{S}_i$ but is in one-to-one correspondence with $\mathbf{S}_i$.

For algebras over a field we have the following corollary to Theorem 2:

COROLLARY. *Let* $\mathbf{A}$ *be an algebra over a field* F *and* $\mathbf{B}$ *an ideal of* $\mathbf{A}$. *If* T *is a regular automorphism of* $\mathbf{A}$ *such that* $T(\mathbf{B}) = \mathbf{B}$ *and* T *is of finite degree on* $\mathbf{B}$, *then the induced automorphism* $T^*\colon \mathbf{A}-\mathbf{B} \to \mathbf{A}-\mathbf{B}$ *is regular.*

Proof. The minimal equation of T on $\mathbf{B}$ can be written in the form

$$\alpha_0 E+\alpha_1(T-E)+\alpha_2(T-E)^2+...+\alpha_k(T-E)^k = 0,$$

where $\alpha_0, \alpha_1, \alpha_2,..., \alpha_k$ are elements of F. Since T is regular, $\alpha_0 \neq 0$. Hence, if $x \in \mathbf{B}$, we have $\alpha_0 x \in (T-E)\mathbf{B}$ and so $x \in (T-E)\mathbf{B}$. It follows that $\mathbf{B} \subseteq (T-E)\mathbf{B}$, whence, by Theorem 2, T^* is regular.

4. Applications to rings with radicals

THEOREM 3. *Let* $\mathscr{C}$ *be a class of rings with the properties*:

(i) *if* $\mathbf{R} \in \mathscr{C}$, *then the transformation ring of* $\mathbf{R}$ *satisfies the minimum condition*,

(ii) *if* $\mathbf{R} \in \mathscr{C}$, *then any homomorphic image of* $\mathbf{R}$ *is in* $\mathscr{C}$.

Suppose that $\mathbf{R} \in \mathscr{C}$ *admits a regular automorphism* T *such that* $(T-E)\mathbf{N} = \mathbf{N}$, *where* $\mathbf{N}$ *is the radical of* $\mathbf{R}$. *Then either* $\mathbf{R} = \mathbf{N}$ *or* $\mathscr{C}$ *contains a simple ring admitting a regular automorphism.*

Proof. Every $\mathbf{R}$ of $\mathscr{C}$ contains a radical $\mathbf{N}$ because of (i). It is easily seen that $\mathbf{N}$ is a characteristic ideal, so that $T(\mathbf{N}) = \mathbf{N}$. Suppose that $\mathbf{R} \neq \mathbf{N}$. Then, by Theorem 2, $\mathbf{R}^* = \mathbf{R}-\mathbf{N}$ admits a regular automorphism T^*. Since $\mathbf{R}^*$ is semi-simple and its transformation ring satisfies the minimum condition, we have

$$\mathbf{R}^* = \mathbf{R}_1 \oplus \mathbf{R}_2 \oplus ... \oplus \mathbf{R}_m$$

where $\mathbf{R}_1,..., \mathbf{R}_m$ are simple two-sided ideals of $\mathbf{R}^*$; this decomposition is unique. But T^* is an automorphism of $\mathbf{R}^*$, and so

$$\mathbf{R}^* = T^*(\mathbf{R}_1) \oplus T^*(\mathbf{R}_2) \oplus ... \oplus T^*(\mathbf{R}_m),$$

where, for each $i = 1,..., m$, $T^*(\mathbf{R}_i)$ is a simple two-sided ideal of $\mathbf{R}^*$.

It follows that there exists a permutation $\pi(i)$ of the integers $1,\ldots, m$ such that
$$T^*(\mathbf{R}_i) = \mathbf{R}_{\pi(i)}.$$
Let k be the least positive integer such that
$$T^{*k}(\mathbf{R}_1) = \mathbf{R}_1$$
(that is, the order of the cycle in $\pi(i)$ containing the integer 1). Then $T^{*k}\colon \mathbf{R}_1 \to \mathbf{R}_1$ is an automorphism. Suppose that x in $\mathbf{R}_1$ is such that $T^{*k}(x) = x$. Then
$$y = x+T^*(x)+\ldots+T^{*k-1}(x)$$
is fixed under T^*. But T^* is regular, and so $y = 0$. Therefore, since x, $T^*(x),\ldots, T^{*k-1}(x)$ belong to different direct summands, we have $x = 0$. Hence $T^{*k}\colon \mathbf{R}_1 \to \mathbf{R}_1$ is a regular automorphism.

By condition (ii), $\mathbf{R} \in \mathscr{C}$ implies that $\mathbf{R}_1 \in \mathscr{C}$. It follows that $\mathscr{C}$ contains a simple ring admitting a regular automorphism, and so Theorem 3 is proved.

We apply this result to the cases of associative rings satisfying the minimum condition, and Lie and Jordan algebras of finite dimension.

THEOREM 4. *Let* $\mathbf{R}$ *be an associative ring satisfying the minimum condition and admitting a regular automorphism* T *such that*
$$(T-E)\mathbf{N} = \mathbf{N},$$
where $\mathbf{N}$ *is the radical of* $\mathbf{R}$. *Then* $\mathbf{R}$ *is nilpotent.*

Proof. Let $\mathscr{R}$ be the class of associative rings satisfying the minimum condition. Conditions (i) and (ii) of Theorem 3 are clearly satisfied in $\mathscr{R}$. If a ring $\mathbf{R}$ in $\mathscr{R}$ admits a regular automorphism T such that $(T-E)\mathbf{N} = \mathbf{N}$ and $\mathbf{R} \neq \mathbf{N}$, then, by Theorem 3, $\mathscr{R}$ contains a simple ring admitting a regular automorphism. But a simple associative ring satisfying the minimum condition has an identity [(**6**) 46] and so cannot admit a regular automorphism. Hence $\mathbf{R} = \mathbf{N}$ and therefore $\mathbf{R}$ is nilpotent [(**6**) 38].

THEOREM 5. *Let* $\mathbf{L}$ *be a Lie algebra of finite dimension, over a field of characteristic zero, admitting a regular automorphism. Then* $\mathbf{L}$ *is soluble.*

Proof. Let $\mathscr{L}$ be the class of Lie algebras of finite dimension over a field of characteristic zero. Conditions (i) and (ii) of Theorem 3 are satisfied in $\mathscr{L}$. If $\mathbf{L} \in \mathscr{L}$ and admits a regular automorphism T, then the radical of $\mathbf{L}$ satisfies $(T-E)\mathbf{N} = \mathbf{N}$ since $\mathbf{L}$ is finite-dimensional. If $\mathbf{L} \neq \mathbf{N}$, then, by Theorem 3, $\mathscr{L}$ contains a simple algebra admitting a regular automorphism. But no such algebra exists, as follows from

results proved by Borel and Mostow [(**3**) Propositions 4.2 and 4.3]. Hence $\mathbf{L} = \mathbf{N}$, and so $\mathbf{L}$ is a soluble algebra.

THEOREM 6. *Let* $\mathbf{J}$ *be a Jordan algebra of finite dimension over a field of characteristic other than* 2, *admitting a regular automorphism. Then* $\mathbf{J}$ *is nilpotent.*

Proof. As in the proof of Theorem 5, we can show that, if $\mathbf{J}$ does not coincide with its radical $\mathbf{N}$, then there is a simple Jordan algebra of finite dimension admitting a regular automorphism. But a simple Jordan algebra contains an identity (**2**) and so cannot admit a regular automorphism. It follows that $\mathbf{J} = \mathbf{N}$, and therefore $\mathbf{J}$ is nilpotent, since the radical of a Jordan algebra is nilpotent (**2**).

5. Regular automorphisms without finiteness restrictions

I conclude this paper by showing that very little is preserved from the above results when finiteness restrictions are removed altogether.

Let $\mathbf{R}$ be a ring and let Z be the set of all integers. Let $\mathbf{R}^Z$ be the discrete direct sum consisting of all functions $\alpha\colon Z \to \mathbf{R}$ such that $\alpha(i)$ is zero for all but a finite number of integers i, together with addition and multiplication defined by

$$(\alpha+\beta)(i) = \alpha(i)+\beta(i),$$
$$(\alpha\beta)(i) = \alpha(i)\beta(i).$$

Define $T\colon \mathbf{R}^Z \to \mathbf{R}^Z$ by $\qquad T(\alpha) = \alpha_1,$

where $\qquad \alpha_1(i) = \alpha(i+1).$

Then T is an automorphism of $\mathbf{R}^Z$. If $\alpha\colon Z \to \mathbf{R}$ is zero for all but a finite number of integers i and satisfies $\alpha(i) = \alpha(i+1)$, then clearly $\alpha(i) = 0$ for all i. Hence T is regular. Therefore, given any ring $\mathbf{R}$, the ring $\mathbf{R}^Z$ admits a regular automorphism. In particular, by choosing $\mathbf{R}$ to be a simple associative ring, we have an example of a semi-simple associative ring admitting a regular automorphism.

REFERENCES

1. A. A. Albert, 'The radical of a non-associative algebra', *Bull. American Math. Soc.* 48 (1942) 891–7.

2. —— 'A structure theory for Jordan algebras', *Ann. of Math.* 48 (1947) 546–67.

3. A. Borel and G. D. Mostow, 'On semi-simple automorphisms of Lie algebras', *Ann. of Math.* 61 (1955) 389–405.

4. Graham Higman, 'Groups and rings having automorphisms without non-trivial fixed elements', *J. London Math. Soc.* 32 (1957) 321–34.

5. N. Jacobson, 'A note on automorphisms and derivations of Lie algebras', *Proc. American Math. Soc.* 6 (1955) 281–3.
6. —— *The structure of rings*, American Math. Soc. Colloquium Publications, vol. 37 (1956).
7. W. E. Jenner, 'The radical of a non-associative ring', *Proc. American Math. Soc.* 1 (1950) 348–51.
8. E. M. Patterson, 'Note on non-associative rings with regular automorphisms', *J. London Math. Soc.* 34 (1959) 457–64.

KUMMER'S PRINCIPLE

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1. Introduction

KUMMER's principle or theorem that the hypergeometric differential equation

$$\delta(\delta+c-1)y = x(\delta+a)(\delta+b)y, \tag{1}$$

or equivalently

$$x(1-x)D^2y+\{c-(a+b+1)x\}Dy-aby = 0, \tag{2}$$

has 24 different solutions is well known. Here we examine its extension to the second-order equation of rank two, i.e. an equation of the form

$$f_0(\delta)y+xf_1(\delta)y+x^2f_2(\delta)y = 0, \tag{3}$$

where the f are quadratics in δ.

Briefly Kummer's principle states that there are six types of solution expressible respectively in terms of variables

$$x, \quad \frac{1}{x}, \quad 1-x, \quad \frac{1}{1-x}, \quad 1-\frac{1}{x}, \quad \frac{x}{x-1}$$

(which correspond to a set of cross-ratios), and that there are four varieties of each type. Thus in the variable x we have the 'standard' solution

$$F(a,b;c;x) \equiv \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{n!\,(c)_n} x^n. \tag{4}$$

Euler's identity

$$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x) \tag{5}$$

gives an alternative form of this solution. To these we add the corresponding forms of the 'second' solution,

$$\left.\begin{array}{l} x^{1-c}F(a-c+1,b-c+1;2-c;x) \\ x^{1-c}(1-x)^{c-a-b}F(1-a,1-b;2-c;x) \end{array}\right\} \tag{6}$$

To determine the various domains of co-existence of the several solutions, i.e. the areas where the regions of convergence overlap, is an unrewarding task: we prefer to think of the transformations as showing the 'automorphisms' of the equation itself. From this point of view all the automorphisms are readily deducible from (i) the 'reversing'

substitution $x \to x^{-1}$, which, from (1), preserves the form of the equation and (ii) Euler's other identity

$$F\left(a, b; c; \frac{x}{x-1}\right) = (1-x)^{-a}F(a, c-b; c; x) \tag{7}$$

(within the common region of convergence). This second identity may be regarded as more fundamental than the earlier identity (5) since, exploiting the symmetry in a, b, we can deduce (5) from it.

2. The equation of rank two

The hypergeometric equation (1) has singularities at 0, 1, ∞. We take the analogous equation of rank two to have singularities at 0, p^{-1}, q^{-1}, ∞ and write its equation in either of the equivalent forms

$$\{\delta(\delta+c-1)-px(\delta+a)(\delta+b)-qx(\delta+a')(\delta+b')+ \\ +pqx^2(\delta+e)(\delta+f)\}y = 0, \tag{8}$$

$$x(1-px)(1-qx)D^2y+ \\ +\{c-(a+b+1)px-(a'+b'+1)qx+(e+f+1)pqx^2\}Dy- \\ -(abp+a'b'q-efpqx)y = 0. \tag{9}$$

We should note that, regarding p, q as fixed, we have *seven* parameters a, b, a', b', c, e, f in the equation. This is excessive: for in (8) the term in x has at most a form

$$x\{(p+q)\delta^2+K\delta+K'\},$$

while the terms in 1, x^2 just suffice to determine c, e, f: this gives five independent parameters.

Now to show the behaviour of the hypergeometric equation at its singularities we rewrite (2) in a form

$$D^2y+\left(\frac{c}{x}+\frac{c-a-b-1}{1-x}\right)Dy-\frac{ab}{x(1-x)}y = 0. \tag{10}$$

For (9) we should have a corresponding form

$$D^2y+\left(\frac{c}{x}+\frac{Ap}{1-px}+\frac{A'q}{1-qx}\right)Dy+\frac{B+B'x}{x(1-px)(1-qx)}y = 0. \tag{11}$$

Comparing this with (9) we identify

$$c(1-px)(1-qx)+Apx(1-qx)+A'qx(1-px)$$

with $\quad c-(a+b+1)px-(a'+b'+1)qx+(e+f+1)pqx^2,$

so that

$$\left.\begin{aligned} A = c-a-b-1, \qquad A' = c-a'-b'-1 \\ A+A' = c-e-f-1 \end{aligned}\right\}, \tag{12}$$

whence $\quad a+b+a'+b' = c+e+f-1. \qquad (13)$

This imposes one condition on the seven parameters. We have not yet found a second 'compulsive' condition to reduce the parameters to their proper independent number of five.

We complete the identification of (10), (11) by noting that

$$B = -abp - a'b'q, \qquad B' = efpq. \tag{14}$$

Then the best 'standard' form of (9) or (11) is probably

$$D^2y + \left\{\frac{c}{x} + \frac{(c-a-b-1)p}{1-px} + \frac{(c-a'-b'-1)q}{1-qx}\right\}Dy - \\ - \frac{abp+a'b'q-efpqx}{x(1-px)(1-qx)}y = 0. \tag{15}$$

3. Euler's first identity

To find an analogue of Euler's identity (5) we write

$$y = (1-px)^{-\lambda}z,$$

for some constant λ, so that operating on z we replace D by

$$D + \frac{p\lambda}{1-px}.$$

Then (11) gives

$$\left(D+\frac{p\lambda}{1-px}\right)^2 z + \left(\frac{c}{x}+\frac{Ap}{1-px}+\frac{A'q}{1-qx}\right)\left(D+\frac{p\lambda}{1-px}\right)z + \\ + \frac{B+B'x}{x(1-px)(1-qx)}z = 0,$$

i.e.

$$D^2z + \left\{\frac{c}{x}+\frac{(A+2\lambda)p}{1-px}+\frac{A'q}{1-qx}\right\}Dz + \\ + \left\{\frac{\lambda(\lambda+A+1)p^2}{(1-px)^2} + \frac{c\lambda p(1-qx)+A'\lambda pqx+B+B'x}{x(1-px)(1-qx)}\right\}z = 0. \tag{16}$$

To maintain the form of (11) we must remove the term in $(1-px)^{-2}$ by taking $\lambda = -(A+1) = a+b-c$, from (12). Then, on reduction and substitution from (14), and using (13), we can write (15) in the convenient form

$$D^2z + \left\{\frac{c}{x}+\frac{(c-a-b+2\lambda-1)p}{1-px}+\frac{(c-a'-b'-1)q}{1-qx}\right\}Dz - \\ - \frac{(a-\lambda)(b-\lambda)p+a'b'q-(e-\lambda)(f-\lambda)pqx}{x(1-px)(1-qx)}z = 0.$$

Thus a', b', c are unaltered and a, b, e, f become respectively $a-\lambda$, $b-\lambda$, $e-\lambda$, $f-\lambda$, where we notice that the identity (13) is preserved.

We need a notation analogous to $F(a, b; c; x)$ to represent the

'standard' solution defined as that which ascends in powers of x from a leading term unity. For this we write

$$V\begin{bmatrix} a,b;\,a',b'; \\ c,e,f \end{bmatrix}px,qx\Bigg] = 1+\frac{ab}{c}px+\frac{a'b'}{c}qx+\dots.$$

Evidently $V = F(a,b;c;px)$ when $q = 0$ and $V = F(a',b';c;qx)$ when $p = 0$. As we see later in § 4.3, V also reduces to a form in F when $q = p$.

This may not be the best notation to use but it has been found sufficiently convenient. The justification for grouping c, e, f together in the lower line is that, if we suppose a solution of the differential equation (8) in the form

$$y = \sum_{n=0}^{\infty}\frac{K_n}{n!\,(c)_n}x^n,$$

then substitution and reduction gives a recurrence relation

$$K_{n+1}-\{(a+n)(b+n)p+(a'+n)(b'+n)q\}K_n+$$
$$+n(c+n-1)(e+n-1)(f+n-1)pqK_{n-1} = 0,$$

so that K_n is symmetrical in c, e, f. This symmetry appears also in the identity (13).

Then in terms of this notation we can express what we have just proved, namely the analogue of Euler's identity (5), in the form

$$V\left[\begin{matrix} a,b;\,a',b'; \\ c,e,f \end{matrix}\;px,qx\right]$$
$$= (1-px)^{-\lambda}V\left[\begin{matrix} a-\lambda,b-\lambda;\,a',b'; \\ c,e-\lambda,f-\lambda \end{matrix}\;px,qx\right] \quad (\lambda = a+b-c)$$
$$= (1-px)^{c-a-b}V\left[\begin{matrix} c-b,c-a;\,a',b'; \\ c,c+e-a-b,c+f-a-b; \end{matrix}\;px,qx\right], \tag{17}$$

This, not unnaturally, reduces to (5) when $q = 0$.

There is, of course, the similar identity in q, namely

$$V\left[\begin{matrix} a,b;\,a',b'; \\ c,e,f \end{matrix}\;px,qx\right]$$
$$= (1-qx)^{c-a'-b'}V\left[\begin{matrix} a,b;\,c-a',c-b'; \\ c,c+e-a'-b',c+f-a'-b'; \end{matrix}\;px,qx\right]. \tag{18}$$

Combining these two transformations and remembering (13) we have

$$V\left[\begin{matrix} a,b;\,a',b'; \\ c,e,f \end{matrix}\;px,qx\right]$$
$$= (1-px)^{c-a-b}(1-qx)^{c-a'-b'}V\left[\begin{matrix} c-a,c-b;\,c-a',c-b'; \\ c,c-f+1,c-e+1 \end{matrix}\;px,qx\right]. \tag{19}$$

These identities extend the standard solution V into four solutions in which the variables are px, qx.

4. Euler's second identity

4.1. To obtain the analogue of Euler's identity (7) we impose on the equation (16) the substitution $x \to x_1$, where equivalently

$$x_1 = \frac{x}{px-1}, \qquad (1-px)(1-px_1) = 1, \tag{20}$$

and so, where $D_1 \equiv d/dx_1$,

$$D = -(1-px_1)^2 D_1, \qquad D^2 = (1-px_1)^4 D_1^2 - 2p(1-px_1)^3 D_1,$$

$$1-qx = \frac{1-(p-q)x_1}{1-px_1}.$$

Write further

$$q_1 = p-q. \tag{21}$$

These give, after some reduction,

$$D_1^2 z - \frac{1}{1-px_1}\left\{-\frac{c}{x_1}+(A+2\lambda+2)p+\frac{A'q}{1-q_1 x_1}\right\} D_1 z +$$
$$+\frac{1}{1-px_1}\left\{\frac{\lambda(\lambda+A+1)p^2}{1-px_1}-\frac{B+c\lambda p}{x_1(1-q_1 x_1)}+\frac{(A'-c)\lambda pq+B'}{(1-px_1)(1-q_1 x_1)}\right\} z = 0. \tag{22}$$

Here in the coefficient of z we have, from (12), (14), (21),

$$\frac{(A'-c)\lambda pq+B'}{(1-px_1)(1-q_1 x_1)} = \frac{\{ef-(e+f+A+1)\lambda\}p(p-q_1)}{(1-px_1)(1-q_1 x_1)}$$
$$= \{ef-(e+f+A+1)\lambda\}\left(\frac{p^2}{1-px_1}-\frac{pq_1}{1-q_1 x_1}\right).$$

Then, on further reduction, the equation giving z can be written

$$D_1^2 z + \left\{\frac{c}{x_1}+\frac{e+f-2\lambda-1}{1-px_1}+\frac{c-a'-b'-1}{1-q_1 x_1}\right\} D_1 z + \frac{\{\lambda^2-(e+f)\lambda+ef\}p^2}{(1-px_1)^2} z -$$
$$-\frac{(c\lambda-ab-a'b')p+a'b'q_1+\{ef-(e+f+c-a-b)\lambda\}pq_1 x_1}{x_1(1-px_1)(1-q_1 x_1)} z = 0.$$

As usual we choose λ to remove the term in $(1-px_1)^{-2}$ and so preserve the form of the differential equation. Thus we take $\lambda = e$ or f; with $\lambda = e$ the equation has the form

$$D^2 z_1 + \left\{\frac{c}{x_1}+\frac{f-e-1}{1-px_1}+\frac{c-a'-b'-1}{1-q_1 x_1}\right\} D_1 z -$$
$$-\frac{(ce-ab-a'b')p+a'b'q_1-e(c+e-a-b)pq_1 x_1}{x_1(1-px_1)(1-q_1 x_1)} z. \tag{23}$$

This is the equation whose standard solution is

$$z = V\begin{bmatrix} a_1, b_1; a', b'; \\ c, e, c+e-a-b; \end{bmatrix} px_1, q_1 x_1$$

when the parameters a_1, b_1 are such that

$$f-e = c-a_1-b_1, \qquad ce-ab-a'b' = a_1 b_1,$$

i.e.
$$a_1+b_1 = c+e-f, \qquad ab+a'b'+a_1 b_1 = ce. \tag{24}$$

For symmetry in e, f it is better to work with new parameters

$$a'' = e-a_1, \qquad b'' = e-b_1,$$

for these are given by

$$a''+b'' = e+f-c, \qquad ab+a'b'+a''b'' = ef. \tag{25}$$

Putting these together we have the analogue of Euler's second identity in the form

$$V\begin{bmatrix} a, b; a', b'; \\ c, e, f \end{bmatrix} px, qx = (1-px)^{-e} V\begin{bmatrix} e-a'', e-b''; a', b'; \\ c, e, c+e-a-b \end{bmatrix} \frac{px}{px-1}, \frac{(p-q)x}{px-1}. \tag{26}$$

In this we can interchange p with q or e with f or both, thus getting three variants of the identity. Interchange of e and f gives us

$$V\begin{bmatrix} a, b; a', b'; \\ c, e, f \end{bmatrix} px, qx = (1-px)^{-f} V\begin{bmatrix} f-a'', f-b''; a', b'; \\ c, c+f-a-b, f \end{bmatrix} \frac{px}{px-1}, \frac{(p-q)x}{px-1}. \tag{27}$$

4.2. As with the ordinary hypergeometric function $F(a, b; c; x)$, comparison of (26), (27) gives (17), the analogue of Euler's first identity. Precisely we make in (27) the substitution

$$a, b, e, f \to a-\lambda, b-\lambda, f-\lambda, e-\lambda,$$

where $\lambda = a+b-c$. This entails also

$$a'', b'' \to a''-\lambda, b''-\lambda$$

and replaces (27) by

$$V\begin{bmatrix} a-\lambda, b-\lambda; a', b'; \\ c, e-\lambda, f-\lambda \end{bmatrix} px, qx$$

$$= (1-px)^{-e+\lambda} V\begin{bmatrix} e-a'', e-b''; a', b'; \\ c, e, c+e-a-b \end{bmatrix} \frac{px}{px-1}, \frac{(p-q)x}{px-1}.$$

Comparison of this with (26) establishes (17).

4.3. Two characteristics of the equation emerge from the identity (26). In the first place the transformation of the parameters a, b is quadratic: in other words a'', b'' are defined irrationally in terms of the original parameters. There still remains, of course, one degree of

freedom among these parameters, but it does not seem possible to employ it usefully by imposing a condition of rationality on a'', b''. It appears better to accept the quadratic relation as part of the essential structure of the equation, which then seems to have a 'triangular' shape. There are three pairs of 'upper' parameters a, b; a', b'; a'', b''; and the original p, q must be extended by $p-q$ or $q-p$. In the sequel we find that the association is skew: we can write it (q,a,b), $(-p,a',b')$, $(p-q,a'',b'')$. Finally, these three sets are found to be fully sufficient for exploring the transformations of the equation.

4.4. If in (26) we put $q = p$, we get the identity

$$V\begin{bmatrix} a,b;\,a',b'; \\ c,e,f \end{bmatrix} px,px\Big] = (1-px)^{-e}V\begin{bmatrix} e-a'',e-b'';\,a',b'; & \dfrac{px}{px-1},0 \\ c,e,c+e-a-b & \end{bmatrix}$$

$$= (1-px)^{-e}F\left[e-a'',e-b'';\,c;\frac{px}{px-1}\right]$$

$$= (1-px)^{-a''}F(e-a'',c-e+b'';\,c;\,px), \tag{27 a}$$

by Euler's identity (7). This is the third case, mentioned in § 3 above, in which V is reducible in terms of F. The asymmetry is only apparent since Euler's first identity shows it to be equal to the same form with a'', b'' interchanged.

We can, of course, prove directly that (27 a) satisfies the equation for V when $q = p$. For substitution of $(1-x)^d y$ into the ordinary hypergeometric differential equation shows that

$$y = (1-x)^{-d}F(a,b;\,c;\,x)$$

satisfies the three-term differential equation

$$\delta(\delta+c-1)-x\{2\delta^2+(a+b+c+2d-1)\delta+ab+cd\}+$$
$$+x^2(\delta+a+d)(\delta+b+d)y = 0.$$

We have then only to give a, b, d the appropriate values and to recall the identities connecting the parameters.

5. The group of variables

5.1. We consider now transformations of the variables in V ignoring problems of convergence: we, in fact, now regard V as indicating the differential equation itself and consider automorphisms of the equation. We have seen in § 3 that the analogue of Euler's first identity gives four variants in unchanged variables px, qx. We obtain a second and distinct solution of the differential equation (as with the ordinary

hypergeometric equation) by writing in it $y = x^{1-c}z$. This second solution is

$$y = x^{1-c}V\begin{bmatrix} a-c+1, b-c+1; a'-c+1, b'-c+1; \\ 2-c, e-c+1, f-c+1 \end{bmatrix} px, qx. \quad (28)$$

This again has its four 'Euler' variants and we have in all eight forms in variables px, qx. Write ξ, η for px, qx. Then the transformation (26) changes the variables from ξ, η to $\xi/(\xi-1)$, $(\xi-\eta)/(\xi-1)$. Denote this transformation by P and write

$$P(\xi,\eta) = \left(\frac{\xi}{\xi-1}, \frac{\xi-\eta}{\xi-1}\right). \quad (29)$$

Q, the similar transformation in q, will give

$$Q(\xi,\eta) = \left(\frac{\eta-\xi}{\eta-1}, \frac{\eta}{\eta-1}\right). \quad (30)$$

There are corresponding transformations, P' and Q' say, in which f replaces e. We need not separately consider these since $P'P$ restores the original variables ξ, η and, in fact, as we have seen, gives an 'Euler variant'.

5.2. We must, however, consider the 'reversing' transformation, R say. Here we write $y = x^{-e}z$ in the equation (8) and make the substitution $x \to 1/pqx$. This changes (8) into

$$\{\delta(\delta+e-f)-px(\delta+e-a)(\delta+e-b)-qx(\delta+e-a')(\delta+e-b')+ \\ +pqx^2(\delta+e)(\delta+e-c+1)\}z = 0, \quad (31)$$

an equation of the same form. There is, of course, an analogous form in which e, f interchange.

Evidently $$R(\xi,\eta) = (\eta^{-1}, \xi^{-1}). \quad (32)$$

We should, perhaps, include the transformation C which interchanges p, q with their associated parameters, so that

$$C(\xi,\eta) = (\eta,\xi). \quad (33)$$

This, of course, in view of the symmetry is a mere change of notation and not of substance.

These substitutions P, Q, R, C are all reversible, so that their squares are each the identity. It remains to consider what variety of transformation can be got from their various products.

Applying (30) to (29) we get, on reduction,

$$QP(\xi,\eta) = \left(\frac{\eta}{\eta-1}, \frac{\eta-\xi}{\eta-1}\right),$$

which is just Q, or strictly CQ. Similarly PQ is effectively P. This exhausts the possibilities involving P, Q alone, and gives nothing new.

Applying (32) to (29) and (30) we get, on reduction

$$RP(\xi,\eta)=\left(\frac{\xi-1}{\xi-\eta},\frac{\xi-1}{\xi}\right),\qquad RQ(\xi,\eta)=\left(\frac{\eta-1}{\eta},\frac{\eta-1}{\eta-\xi}\right),\tag{34}$$

and, in reverse,

$$PR(\xi,\eta)=\left(\frac{1}{1-\eta},\frac{\xi-\eta}{\xi(1-\eta)}\right),\qquad QR(\xi,\eta)=\left(\frac{\eta-\xi}{\eta(1-\xi)},\frac{1}{1-\xi}\right).\tag{35}$$

These are all different and give new forms of the variables; they exhaust the double products.

5.3. Of the triple products we have, on reduction

$$RPR(\xi,\eta)=\left(\frac{\xi(1-\eta)}{\xi-\eta},1-\eta\right),\qquad RQR(\xi,\eta)=\left(1-\xi,\frac{\eta(1-\xi)}{\eta-\xi}\right),\tag{36}$$

$$PRP(\xi,\eta)=\left(\frac{\xi-1}{\eta-1},\frac{\eta(\xi-1)}{\xi(\eta-1)}\right),\qquad QRQ(\xi,\eta)=\left(\frac{\xi(\eta-1)}{\eta(\xi-1)},\frac{\eta-1}{\xi-1}\right).\tag{37}$$

But
$$PRQ(\xi,\eta)=\left(1-\eta,\frac{\xi(1-\eta)}{\eta(\xi-\eta)}\right),$$

so that PRQ is equivalent to RPR, and similarly QRP is equivalent to RQR. This exhausts the triple products.

5.4. Continued products, if they are to be irreducible, must be composed of a run of R with each pair separated by P or Q. By what has just been seen a product $PRQR$ is equivalent to RPR^2, i.e. to RP, and similarly $RPRQ$ is equivalent to PR. This leaves only such continued products as $PRPR$, $RPRP$. From (37),

$$PRPR(\xi,\eta)=\left(\frac{\xi(\eta-1)}{\eta(\xi-1)},\frac{\eta-1}{\xi-1}\right),$$

so that $PRPR$ is equivalent to QRQ. Similarly, again from (37),

$$RPRP=\left(\frac{\xi(\eta-1)}{\eta(\xi-1)},\frac{\eta-1}{\xi-1}\right),$$

so that $RPRP$ is again equivalent to QRQ, and now all the four-factor products have been proved reducible, and we need proceed no further. This leaves twelve independent transformations including the identity, namely,

$$1,\ P,\ Q,\ R;\ PR,\ QR,\ RP,\ RQ;\ PRP,\ QRQ,\ RPR,\ RQR.$$

The corresponding pairs of variables are

$$(px, qx), \quad \left(\frac{px}{px-1}, \frac{(p-q)x}{px-1}\right), \quad \left(\frac{(q-p)x}{qx-1}, \frac{qx}{qx-1}\right), \quad \left(\frac{1}{qx}, \frac{1}{px}\right);$$

$$\left(\frac{1}{1-qx}, \frac{p-q}{p(1-qx)}\right), \quad \left(\frac{q-p}{q(1-px)}, \frac{1}{1-px}\right),$$

$$\left(\frac{px-1}{(p-q)x}, \frac{px-1}{px}\right), \quad \left(\frac{qx-1}{qx}, \frac{qx-1}{(q-p)x}\right);$$

$$\left(\frac{px-1}{qx-1}, \frac{q(px-1)}{p(qx-1)}\right), \quad \left(\frac{p(qx-1)}{q(px-1)}, \frac{qx-1}{px-1}\right),$$

$$\left(\frac{p(1-qx)}{p-q}, 1-qx\right), \quad \left(1-px, \frac{q(1-px)}{q-p}\right).$$

With the 'second' solutions and the 'Euler variants' each of these pairs gives rise to eight different forms. There are thus in all 96 different forms of solution.

5.5. Professor J. L. Burchnall has suggested that we ought to look at the group character of these transformations and that we can study them by their permutation of the singularities of the equation.

If we write these singularities in the order $(\infty, p^{-1}, q^{-1}, 0)$ and number them thus as (1 2 3 4), then P, Q, R effect the respective permutations (1 2), (1 3), (1 4)(2 3). The permutation (2 3) by itself is excluded since it is merely the symmetrical interchange C. This exclusion halves the total number of 24 permutations on the four singularities leaving us with 12, which suggests that our number is complete.

6. The full solutions

6.1. In the preceding section we have examined the Kummer transformations in skeleton in terms of the variables alone. These need to be embodied in the full forms of solution. In (17), (18), (19) we already have the 'Euler variants' of V in full; in (28) we have the 'standard' second solution of which the 'Euler variants' are readily written down. The effect of the transformation P is given in (26).

From the symmetry of the equation the transformations fall into symmetrical pairs P, Q; PR, QR, etc. We need give only the form in P; the symmetrical fellow comes, of course, by interchanging p, a, b with q, a', b'. Thus we need consider only the transformations PR, RP,

PRP, *RPR* and, of course, *R* itself. For *R* we have at once from § 5.2, and in particular, from (31),

$$V_R \equiv x^{-e}V\begin{bmatrix} e-a, e-b; e-a', e-b'; & \dfrac{1}{qx}, \dfrac{1}{px} \\ e-f+1, e, e-c+1 & \end{bmatrix}. \tag{38}$$

We may regard this as the 'first' solution and the solution with e, f interchanged as the second solution.

When we apply the Euler transformation P to the function V_R in (38) we need the analogues of a'', b'' for its parameters. Writing α, β for these analogues we have, from (25),

$$\alpha+\beta = e+f-c,$$

$$(e-a)(e-b)+(e-a')(e-b')+\alpha\beta = e(e-c+1),$$

so that $$ab+a'b'+\alpha\beta = ef,$$

and α, β are precisely a'', b''. Thus, from (26), (38),

$$V_R = (1-qx)^{-e}V\begin{bmatrix} e-a'', e-b''; e-a', e-b'; & \dfrac{1}{1-qx}, \dfrac{p-q}{p(1-qx)} \\ e-f+1, e, a+b-f+1 & \end{bmatrix}. \tag{39}$$

This represents the transformation *PR*. Conversely, when we apply *R* to (26) we get the solution

$$V_{RP} = x^{-e}V\begin{bmatrix} a'', b''; e-a', e-b'; & \dfrac{px-1}{(p-q)x}, \dfrac{px-1}{px} \\ a+b-c+1, e, e-c+1; & \end{bmatrix}. \tag{40}$$

6.2. In applying the transformation P to V_{RP} we need to find the analogues of a'', b'', say α, β. Then

$$\alpha+\beta = 2e-a-b, \qquad a''b''+(e-a')(e-b')+\alpha\beta = e(e-c+1),$$

which gives $\alpha, \beta = e-a, e-b$, and we have

$$V_{PRP} = (1-qx)^{-e}V\begin{bmatrix} a, b; e-a', e-b'; & \dfrac{px-1}{qx-1}, \dfrac{q(px-1)}{p(qx-1)} \\ a+b-c+1, e, a+b-f+1; & \end{bmatrix}. \tag{41}$$

Finally, operating on (39) with R from (38) we get

$$V_{RPR} = V\begin{bmatrix} a'', b''; a', b'; & \dfrac{p(1-qx)}{p-q}, 1-qx \\ a'+b'-c+1, e, f; & \end{bmatrix}. \tag{42}$$

This completes the list of the full forms with parameters; it may be remarked that they all pivot on an unvaried parameter e.

THE BEHAVIOUR OF A FUNCTION NEAR AN ISOLATED ESSENTIAL POINT AND THAT OF ITS COEFFICIENT FUNCTION NEAR INFINITY

By R. WILSON (*Swansea*)

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1. Introduction

LET
$$\psi(z) = \sum_0^\infty c_n z^n = G\left(\frac{1}{1-z}\right), \tag{1}$$
where $G(z)$ is an integral function of type h (>0) and positive order ρ. Then the coefficients c_n ($n \geqslant 1$) can be interpolated by a unique integral function $F(z)$ of order $\rho' = \rho/(\rho+1)$ and type $h' = \{(\rho+1)/\rho\}(h\rho)^{1/(\rho+1)}$.

Macintyre and Wilson (**3**) have proved the theorem:

THEOREM 1. *The directions of strongest growth of $F(z)$ as z tends to infinity are the same as those of $\psi(z)$ as z tends to unity.*

By this is meant that
$$\limsup_{R\to\infty} \frac{1}{R^{\rho/(\rho+1)}} \log|F(Re^{i\phi})|$$
$$\text{and} \quad \limsup_{R\to\infty} \frac{1}{R^\rho} \log|G(Re^{i\phi})|, \quad \text{i.e.} \quad \limsup_{r\to 0} r^\rho \log|\psi(1-re^{-i\phi})| \tag{2}$$
reach their maxima for the same values of ϕ.

This is a fairly deep result, and the question arises as to whether and how it can be sharpened. It is proposed to do this in two ways: firstly, to consider more closely the relationship between corresponding directions of strongest growth of $F(z)$ and $G(z)$ and, secondly, to consider the relationship, if any, between the sets of points at which $\log|F(z)|$ and $\log|G(z)|$ are large.

If we set
$$F(z) = \sum_0^\infty \frac{a_n z^n}{n!}, \qquad f(z) = \sum_0^\infty a_n z^{-n-1},$$
then $\psi(e^{-z})$ and $f(z)$ differ by a function which is regular in the neighbourhood of $z = 0$ [cf. (**2**) 3; (**3**) 302]. Since [(**3**) § 3]
$$\limsup_{r\to 0} r^\rho \log|\psi(e^{-re^{-i\phi}})| = \limsup_{r\to 0} r^\rho \log|\psi(1-re^{-i\phi})|$$
$$\text{and} \quad \limsup_{r\to 0} r^\rho \log\left|G\left(\frac{1}{1-e^{-re^{-i\phi}}}\right)\right| = \limsup_{R\to\infty} \frac{1}{R^\rho} \log|G(Re^{i\phi})| \tag{3}$$

it follows from (2) and (3) that we are really concerned with the relationship between the directions of strongest growth of $F(z)$ and $z^{-1}f(z^{-1})$.

These are two integral functions of which the latter is the Laplace transform of the former. In other words, $F(z)$ is the Hadamard product of e^z and $z^{-1}f(z^{-1})$. It is known, however, that the Hadamard product of the sole direction of strongest growth of e^z and a direction of strongest growth of any kind of a second function effectively gives rise to a direction of strongest growth in the composition function, a fact which is put in evidence by the result of Theorem 1 [(**6**) Theorem 2 (i)]. What now has to be determined is whether each resultant direction of strongest growth is of the same nature as the original one. To this end we consider the behaviour under Hadamard composition of the singular directions which arise from the generalized Laplace transforms [(**6**) § 2] of the functions concerned.

2. Directions of strongest growth and singular directions

There is a well-known theorem connecting the directions of strongest growth of an integral function with the singular directions of its generalized Laplace transform [(**6**) Theorem A, where further references are given].

THEOREM A. *The indicator $h(\theta)$ of an integral function of finite order and mean type h attains its maximum h for a direction* $\arg z = \theta$ *if and only if the conjugate direction of its generalized Laplace transform* $\arg z = -\theta$ *is a singular direction.*

The directions of strongest growth are those for which $h(\theta)$ attains its maximum, and singular directions are lines from the origin passing through a singular point on the circle of convergence.

We now proceed to consider the relations between the singular directions of the generalized Laplace transforms of $F(z)$ and $z^{-1}f(z^{-1})$. The generalized Laplace transforms of $F(z)$ and $z^{-1}f(z^{-1})$ are, with $\sigma = \rho^{-1}$,

$$\sum_0^\infty \frac{\Gamma\{(n+1)(\sigma+1)\}}{\Gamma(n+1)} a_n z^{-n-1} \quad \text{and} \quad \sum_0^\infty \Gamma(n\sigma+\sigma) a_n z^{-n-1} \tag{4}$$

respectively, and the latter is the Hadamard product of the former and the series

$$\sum_0^\infty \frac{\Gamma(n\sigma+\sigma)\Gamma(n+1)}{\Gamma\{(n+1)(\sigma+1)\}} z^{-n-1}.$$

This series has an isolated critical point at

$$z = \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}$$

with a dominant algebraic element of weight $[\frac{1}{2}, 0]$ and no other singularities.† Although this critical point is of a more complicated character than an algebraic-logarithmic point, yet its coefficient properties are simpler and the arguments used in the case of an algebraic-logarithmic singularity with a single dominant element (**7**) are readily applicable to this case. Hence we have the lemma:

LEMMA 1. *The Hadamard product of an isolated critical point with a dominant algebraic element and any other kind of singular point, whether isolated with respect to the circumference or an end-point or interior point of a singular arc, is effectively singular.*

Given the component singularity we now have to consider the nature of the resultant singularity. Since the critical singularity is at an isolated point, it cannot change the character of the other component except in regard to multiformity. For the singularities in Lemma 2 are each defined by their relation to neighbouring singularities and, since these remain unchanged by the Hadamard composition referred to above, the character of such singularities is unaltered by the operation. In fact, under Hadamard composition with such a singularity, the star-domain of the composition function is identical with that of the original function. The detailed results are stated explicitly in the lemma:

LEMMA 2. *The Hadamard product of a critical singularity having a dominant algebraic element and* (i) *a unique singularity,* (ii) *an easily approachable singularity,* (iii) *a virtually isolated singularity,* (iv) *an isolable singularity,* (v) *the end-point of a singular arc,* (vi) *a singular interior point of a singular arc, is a singular point of the same kind except that the property of uniformity is not preserved.*

For example, the Hadamard product of the critical point with an isolated essential point is an isolated critical point.

In conformity with a theory recently developed (**5**, **6**), we can restate the results of Lemma 2 in terms of the directions of strongest growth of $F(z)$ and $z^{-1}f(z^{-1})$. In this, the singular directions of the generalized Laplace transforms are replaced by the corresponding directions of strongest growth of the original integral functions. To the categories of singularity referred to in Lemma 2 correspond unique, accessible, isolated, isolable, and bounding or interior directions of a set of directions of strongest growth. Thus, using (**4**) with Theorem A and Lemma 2, we get the theorem:

† From the properties of the integral in [(**3**) 301 (9)] with $\sigma_2 = 1$ and the use of (14) in (**6**).

THEOREM 2. *The corresponding directions of strongest growth of $F(z)$ and $G(z)$ are each of the same kind.*

For, since $\psi(e^{-z})$ and $f(z)$ differ by a function which is regular near $z = 0$, it follows from Theorem 1 and the argument above that the directions of strongest growth of $G(z)$ and $z^{-1}f(z^{-1})$ are of the same kind.

3. Points at which $|F(z)|$ and $|G(z)|$ are large

Along a direction of strongest growth there is a set of points of positive upper linear density for which, at the same time [(**5**) 409]

$$R^{-\rho}\log|G(Re^{i\theta})| > h-\epsilon, \qquad R^{-\rho}\log M(R) > h-\epsilon', \tag{5}$$

for some $\epsilon' \geqslant \epsilon > 0$, where $M(R)$ is the maximum modulus of $G(z)$ on $|z| = R$.

If there is only one direction of strongest growth, then $\log|G(z)|$ is large in the sense given in (5) only at a set of points on this line. In particular, for the function e^z,

$$R^{-1}\log|\exp(Re^{i\theta})| = 1, \qquad R^{-1}\log M(R) = 1 \tag{6}$$

at every point on the positive real axis.

We consider here the Hadamard product of $z^{-1}f(z^{-1})$ and e^z

$$F(z) = \frac{1}{2\pi i}\int\limits_{|u|=R} u^{-1}f(u^{-1})e^{z/u}\frac{du}{u}, \tag{7}$$

which is the inverse Laplace transform of $z^{-1}f(z^{-1})$.

To begin with, only *unique* directions of strongest growth will be considered. Since, in this case, it is possible to dissect the function into two parts, one of which has this as its sole direction of strongest growth while the other has the remaining directions of strongest growth and no others, there is no loss of generality in assuming that $z^{-1}f(z^{-1})$ has only one direction of strongest growth, $\arg z = \theta$, say.

Now suppose that the inequality in (5),

$$\log|u^{-1}f(u^{-1})| > (h-\epsilon)R^{\rho},$$

is satisfied at a set of points $(Re^{i\theta})$ along $\arg u = \theta$. Since, as in (6),

$$\log|e^{z/u}| = R'$$

at all real points $z/u = R'$ (> 0), it follows that at points

$$z = u\frac{z}{u} = RR'e^{i\theta},$$

for $\epsilon > 0$ and R sufficiently large,

$$\log|u^{-1}f(u^{-1})e^{z/u}| > (h-\epsilon)R^{\rho}+R'. \tag{8}$$

We choose a subset (R') to satisfy the relation

$$h\rho R^{\rho} = R',$$

so that $$|z| = RR' = h\rho R^{\rho+1}, \qquad h\rho R^{\rho} = (h\rho)^{1/(\rho+1)}|z|^{\rho/(\rho+1)}, \tag{9}$$

whence $$hR^{\rho}+R' = \frac{\rho+1}{\rho}(h\rho)^{1/(\rho+1)}|z|^{\rho/(\rho+1)}.$$

Thus, the inequality (8) becomes, for some $\epsilon' > 0$,

$$\log|u^{-1}f(u^{-1})e^{z/u}| > (h'-\epsilon')|z|^{\rho'}, \tag{10}$$

where ρ', h' are the respective order and type of $F(z)$ as in § 1.

Assuming, as stated, that $\arg z = \theta$ is the only direction of strongest growth, then (10) will be attained in the integral (7) only for a relatively short length of arc of the circle $|u| = R$. On account of the continuity of the integrand, the method of steepest descent then shows that, for some $\epsilon'' > 0$,

$$|F(z)| > \exp(h'-\epsilon'')|z|^{\rho'} \tag{11}$$

for that set of values of $|z|$ along $\arg z = e^{i\theta}$ satisfying (9), which can be written

$$h'\rho'|z|^{\rho'} = h\rho R^{\rho}, \quad \text{i.e.} \quad |z| = h\rho R^{\rho+1}. \tag{12}$$

Formula (12) thus establishes the relationship between the set of points $Re^{i\theta}$ along a unique direction of strongest growth of $z^{-1}f(z^{-1})$ or $G(z)$ at which the moduli of these functions are large in the sense of (5) and the set of points $|z|e^{i\theta}$ along the corresponding direction of strongest growth of $F(z)$ at which $|F(z)|$ satisfies (11). Hence we have the theorem:

THEOREM 3. *Along corresponding directions of strongest growth of $F(z)$ and $G(z)$ which are unique, the sets of points at which $|F(z)|$ and $|G(z)|$ are large are related by* (12).

There are two kinds of isolable directions of strongest growth, one which is unique and one which is the sole limiting direction of a countable set of directions of strongest growth. In the latter case a dissection process can be used [see (**5**) where further references are given] which is similar to that arising with a unique direction of strongest growth but involving a limiting process. Hence, by a slight modification of the proof, the results of Theorem 3 can be made to apply to this case also. We therefore have the theorem:

THEOREM 4. *Along corresponding directions of strongest growth of $F(z)$ and $G(z)$ which are isolable, the sets of points at which $|F(z)|$ and $|G(z)|$ are large are related by* (12).

4. Sets of directions of strongest growth

Apart from the case referred to above, sets of directions of strongest growth present a more complicated situation than that arising with a unique direction of strongest growth, and one cannot expect such sharp results.

Suppose that the directions of strongest growth of $z^{-1}f(z^{-1})$ are all contained in the angle $|\arg z| \leqslant \alpha$. Then the inequality

$$\log M(r) > (h-\epsilon)r^{\rho} \tag{13}$$

for ϵ (> 0) given arbitrarily, is attained at points in the angle for a set of r of maximal linear density $(\pi-\alpha)/\pi$ at least [(5) Theorem 5].

The proof of this result depends on the application of Theorem A to a theorem of Pólya's [see (5) Theorem E; (4) 622 Satz IV] and the use of the following lemma, due to Lindelöf [(1) 42–45]:

LEMMA 3. *Whenever*

$$(1-\epsilon)\left(\frac{he\rho}{n}\right)^{1/\rho} < |a_n|^{1/n} < (1+\epsilon)\left(\frac{he\rho}{n}\right)^{1/\rho} \quad (\epsilon > 0),$$

then $$(h-\epsilon)r^{\rho} < \log M(r) < (h+\epsilon)r^{\rho} \quad (n > n_0;\ r > r_0),$$

and conversely. Further, the sequences for which these inequalities hold satisfy the relation

$$n = h\rho r^{\rho}\{1+\epsilon(r)\}, \tag{14}$$

where $\epsilon(r)$ tends to zero with $1/r$.

In the case of $F(z)$, the coefficient a_n is replaced by $a_n/n!$ and the results in (13) and Lemma 3 continue to hold provided that ρ, h and $\log M(r)$ are replaced by ρ', h', and $\log M_1(r')$, where $M_1(r')$ is the maximum modulus of $F(z)$ on $|z| = r'$. The two versions of (14) that so appear are connected through the same sequence of n to give

$$h'\rho' r'^{\rho'}\{1+\epsilon'(r')\} = h\rho r^{\rho}\{1+\epsilon(r)\}, \tag{15}$$

where $\epsilon(r)$ and $\epsilon'(r')$ each tend to zero with $1/r$ and $1/r'$.

The consequences of these results are set out in the theorem:

THEOREM 5. *Let $G(z)$ have a set of directions of strongest growth contained in $|\arg z| \leqslant \alpha$, including the directions $\arg z = \pm\alpha$. Then $F(z)$ has a corresponding set of directions of strongest growth contained in the*

same angle, including the arms of the angle. Moreover, the inequalities

$$\log M(r) > (h-\epsilon)r^{\rho}, \qquad \log M_1(r') > (h'-\epsilon')r'^{\rho'},$$

hold for sets of r and r' of maximal linear density $(\pi-\alpha)/\pi$ at least, and corresponding values of r and r' are connected by (15).

It is assumed, as has been proved by Pólya [(4) 613 Satz I], that such a set of directions of strongest growth is closed and not empty. Also, it has been assumed that the angle containing them can be bisected by the positive real axis. As a rule this is not the case, but there is no loss of generality in the assumption since a simple change of the variable z suffices to bring about this situation.

Theorems 3–5 represent the ultimate development of the ideas in Theorem 1, and the results of Theorems 1–5 are set out briefly below.

The directions of strongest growth of $F(z)$ and $G(z)$ are identical and corresponding members are of the same kind, each to each. Further, the sets of points at which $\log|F(z)|$ and $\log|G(z)|$ are large are given by the correspondences in Theorems 3–5. From this one can see how closely the asymptotic behaviour of the coefficient function $F(z)$ reflects the behaviour of $\psi(z)$ in the neighbourhood of its isolated essential point.

Finally, if, in place of $\psi(z)$, we take a function $\chi(z)$ which has in its field of singularities the isolated essential point given by $G(1/(1-z))$, then, by a simple process of dissection, $\chi(z)$ can be expressed as the sum of $G(1/(1-z))$ and $\chi^*(z)$, where $\chi^*(z)$ is regular at $z = 1$. From this it follows that the results obtained for $\psi(z)$ can be applied in the more general case of $\chi(z)$, as in the theorem:

THEOREM. *If an analytic function has, as its nearest singularity to the origin, an isolated essential point of finite positive order and mean type, then its directions of strongest growth in the neighbourhood of this point are identical with those of its coefficient function. Moreover, corresponding directions of strongest growth are of the same kind, each to each, and the modulus of the function is large in the neighbourhood of the singularity at corresponding sets of points to those at which the modulus of the coefficient function is asymptotically large.*

If the isolated essential point is not the nearest singularity to the origin, then it will be necessary to dissect the coefficient function so as to separate off that part of it which is associated with the isolated essential point. The relations between this and the isolated essential point then hold as given in the theorem above.

REFERENCES

1. E. Lindelöf, *Acta Soc. Sci. Fennicae,* 31 (1902) 1–79.
2. A. J. Macintyre, *Proc. London Math. Soc.* (2) 45 (1939) 1–20.
3. A. J. Macintyre and R. Wilson, *J. London Math. Soc.* 22 (1947) 298–304.
4. G. Pólya, *Math. Z.* 29 (1929) 549–640.
5. R. Wilson, *J. London Math. Soc.* 32 (1957) 409–20.
6. —— ibid. 32 (1957) 421–9.
7. —— *Quart. J. of Math.* (Oxford) (2) 9 (1958) 68–72.

THE DETERMINATION OF THE TRANSMISSION COEFFICIENT

By J. B. McLEOD (*Oxford*)

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1. Let us consider the one-dimensional wave equation

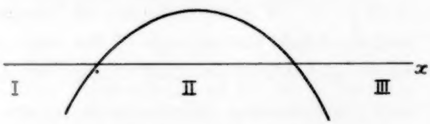
$$\frac{d^2y}{dx^2}+\{\lambda-q(x)\}y = 0 \quad (-\infty < x < \infty) \tag{1.1}$$

where $q(x)$ is a continuous function which tends to $-\infty$ both as $x \to +\infty$ and as $x \to -\infty$. Then, if $q(x)$ satisfies certain conditions of smoothness, Titchmarsh [(1) §§ 5.7, 5.8] has shown that the solutions of (1.1) behave as $x \to +\infty$ like linear combinations of

$$\{\lambda-q(x)\}^{-\frac{1}{4}}\exp\left\{\pm i\int^{x}[\lambda-q(t)]^{\frac{1}{2}}\,dt\right\};$$

and by a similar argument the same result holds as $x \to -\infty$.

The shape of $q(x)$ is roughly as shown:



Since $q(x)$ is proportional to the potential energy, this means that the corresponding physical problem is that of a one-dimensional wave and its penetration (or non-penetration) of a potential energy barrier. Let us suppose that the wave incident upon the barrier is incident from the left. Then in region I there will be both the incident wave and a wave reflected from the barrier, while in region III there can be only a transmitted wave. Since this must be moving to the right, it may be taken in the approximate form (as $x \to +\infty$)

$$\{\lambda-q(x)\}^{-\frac{1}{4}}\exp\{i(w-\tfrac{1}{4}\pi)\}, \tag{1.2}$$

where we now write

$$w = w(x) = \int_{X}^{x}[\lambda-q(t)]^{\frac{1}{2}}\,dt, \tag{1.3}$$

X being the greater zero of $\lambda-q(x)$.

In region I, as already explained, the wave contains both an incident

part (moving to the right) and a reflected part (moving to the left), and so may be taken in the approximate form (as $x \to -\infty$)

$$\{\lambda - q(x)\}^{-\frac{1}{4}}[\alpha \exp\{i(w_1 - \tfrac{1}{4}\pi)\} + \beta \exp\{-i(w_1 - \tfrac{1}{4}\pi)\}], \tag{1.4}$$

where we write $$w_1 = w_1(x) = \int_x^{X_1} [\lambda - q(t)]^{\frac{1}{2}}\, dt, \tag{1.5}$$

X_1 being the lesser zero of $\lambda - q(x)$. In (1.4), the first term gives a wave moving to the left, i.e. the reflected part of the wave, and the second term gives the incident part.

The *transmission coefficient* is in effect the proportion of the incident amplitude which is transmitted and so is defined to be

$$T = 1/\,|\beta|^2.$$

This quantity, being of some physical significance, has been evaluated, and we have, e.g. [(**2**) equation (9.3.104)], that

$$T \simeq e^{-2\kappa}, \tag{1.6}$$

where $$\kappa = \int_{X_1}^{X} |\lambda - q(t)|^{\frac{1}{2}}\, dt. \tag{1.7}$$

The proof given of this is not rigorous, nor is there any estimate of the error involved in (1.6). It is my attempt in this paper to supply a rigorous proof and to obtain an estimate of the error involved.

The assumptions on $q(x)$ which we shall require are as follows:

(i) $q(x)$ is a twice continuously differentiable function of x;

(ii) $q(x) \to -\infty$ as $|x| \to \infty$;

(iii) $q'(x)$ is ultimately negative as $x \to +\infty$ and ultimately positive as $x \to -\infty$;

(iv) as $|x| \to \infty$, $q(x)$ is three times continuously differentiable, with

$$\frac{q'(x)}{q(x)} = O\left(\frac{1}{x}\right), \qquad \frac{q''(x)}{q'(x)} = O\left(\frac{1}{x}\right), \qquad \frac{q'''(x)}{q'(x)} = O\left(\frac{1}{x^2}\right);$$

(v) as $|x| \to \infty$, $q''(x)$ is ultimately of one sign, though this sign may be different for the two different x-limits; if this sign is positive for either limit, then for that limit we must also have

$$q'(x)/q(x) \asymp 1/x.$$

As we shall see, these conditions are sufficient to ensure that there are unique solutions of (1.1) which behave like (1.2), (1.4) as $|x| \to \infty$. We shall prove, under these conditions, that, if the particular solution which behaves as $x \to +\infty$ like (1.2) also behaves as $x \to -\infty$ like (1.4),

then for sufficiently large negative λ,

$$1/|\beta|^2 = e^{-2\kappa}\{1+O(\lambda^{-\frac{3}{2}})+O(\lambda^{-\frac{1}{2}}X^{-1})+O(\lambda^{-\frac{1}{2}}X_1^{-1})\},$$

where, as before, X_1, X are the lesser and greater zeros respectively of $\lambda-q(x)$ and κ is given by (1.7).

2. For $x \geqslant X_1$, we define $w(x)$ by (1.3), together with

$$\arg w = \begin{cases} 0 & (x > X), \\ \frac{3}{2}\pi & (x < X). \end{cases}$$

If we further define
$$\eta(x) = \{\lambda-q(x)\}^{\frac{1}{4}}y,$$
then the equation (1.1) transforms into

$$\frac{d^2\eta}{dw^2}+\left(1+\frac{5}{36w^2}\right)\eta = f(x)\eta, \tag{2.1}$$

where
$$f(x) = \frac{5}{36w^2}-\frac{q''(x)}{4\{\lambda-q(x)\}^2}-\frac{5q'^2(x)}{16\{\lambda-q(x)\}^3}.$$

Further, (2.1) is formally equivalent to the integral equation

$$\eta(x) = (\tfrac{1}{2}\pi w)^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(w)+$$
$$+\tfrac{1}{2}\pi i\int\limits_x^\infty \{H_{\frac{1}{3}}^{(1)}(w)J_{\frac{1}{3}}(\theta)-J_{\frac{1}{3}}(w)H_{\frac{1}{3}}^{(1)}(\theta)\}w^{\frac{1}{2}}\theta^{\frac{1}{2}}f(t)\eta(t)\{\lambda-q(t)\}^{\frac{1}{2}}\,dt, \tag{2.2}$$

where $\theta = w(t)$.

We can now follow Titchmarsh, as in (**3**), in proving that, for sufficiently large λ,

$$\int\limits_0^\infty |f(x)|\,|\lambda-q(x)|^{\frac{1}{2}}\,dx = O(\lambda^{-\frac{3}{2}})+O(\lambda^{-\frac{1}{2}}X^{-1}). \tag{2.2 a}$$

Titchmarsh in the reference given is dealing with the case of large positive λ and $q(x) \to +\infty$ as $x \to +\infty$, with ultimately $q''(x) > 0$, but the analysis is sufficiently similar in the present instance to require no repetition, provided that the ultimate sign of $q''(x)$ is taken to be negative. (The alternative of the positive sign will be considered below.) Also, if $q''(x)$ is ultimately negative, and, since we also have $q'(x)$ ultimately negative, it follows that ultimately $q(x) < -Ax$, for some positive constant A, and so $\lambda^{-\frac{3}{2}}$ is negligible beside $\lambda^{-\frac{1}{2}}X^{-1}$. Thus the term $O(\lambda^{-\frac{3}{2}})$ in (2.2 a) can be omitted. This is in accordance with Titchmarsh's work, where the term $O(\lambda^{-\frac{3}{2}})$ is not mentioned because it is negligible.

There is need to mention the term $O(\lambda^{-\frac{3}{2}})$ only when the ultimate

sign of $q''(x)$ is positive, with of course $q'(x)/q(x) \asymp 1/x$, and the corresponding situation when $q(x) \to +\infty$, i.e. with $q''(x)$ ultimately negative, is not considered by Titchmarsh. However, Titchmarsh's analysis still applies in essence. For the purpose of the assumption that ultimately $q''(x) < 0$—which, as we saw above, reduces our analysis effectively to that given by Titchmarsh—is to ensure that

$$|\lambda - q(\tfrac{1}{2}X)| \geqslant \tfrac{1}{2}|\lambda|. \tag{2.2 b}$$

It would be equally satisfactory for the argument if the constant $\frac{1}{2}$ on the right-hand side of (2.2 b) were replaced by any positive constant A, and we can do this if $q''(x) > 0$, with $q'(x)/q(x) \asymp 1/x$. For then

$$\begin{aligned} \lambda - q(\tfrac{1}{2}X) &= q(X) - q(\tfrac{1}{2}X) \\ &= \tfrac{1}{2}Xq'(\xi) \qquad (\tfrac{1}{2}X < \xi < X) \\ &\asymp \tfrac{1}{2}Xq'(X), \\ &\qquad \text{since } q''(x)/q'(x) = O(1/x) \text{ implies } q'(\tfrac{1}{2}X) \asymp q'(X), \\ &\asymp q(X), \text{ since } q'(x)/q(x) \asymp 1/x, \\ &= \lambda. \end{aligned} \tag{2.2 c}$$

Armed with the result (2.2 a), we can proceed to solve (2.2) by iteration [cf. (4) § 3], and for sufficiently large negative λ the solution satisfies

$$\eta(x) = \begin{cases} (\tfrac{1}{2}\pi w)^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(w) + O(\lambda^{-\frac{1}{6}} e^{-\operatorname{im} w}) + O(\lambda^{-\frac{1}{2}} X^{-1} e^{-\operatorname{im} w}) \\ \qquad\qquad\qquad\qquad \text{(uniformly in } 0 \leqslant x \leqslant X), \\ (\tfrac{1}{2}\pi w)^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(w) + O\left\{\displaystyle\int_x^\infty |f(t)|\,|\lambda - q(t)|^{\frac{1}{2}}\,dt\right\} \\ \qquad\qquad\qquad\qquad \text{(uniformly in } X \leqslant x < \infty). \end{cases}$$

It therefore follows that, as $x \to +\infty$,

$$\eta(x) \sim (\tfrac{1}{2}\pi w)^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(w),$$

and, since for large $|w|$, with $\arg w = 0$,

$$(\tfrac{1}{2}\pi w)^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(w) = e^{i(w - \frac{1}{6}\pi - \frac{1}{4}\pi)}\{1 + O(w^{-1})\}, \tag{2.2 d}$$

we have further that

$$\eta(x) \sim \exp\{i(w - \tfrac{1}{6}\pi - \tfrac{1}{4}\pi)\}.$$

But the particular solution of (1.1) which we are investigating, say $\psi(x)$, satisfies

$$\psi(x) \sim \{\lambda - q(x)\}^{-\frac{1}{4}} \exp\{i(w - \tfrac{1}{4}\pi)\},$$

and so

$$\psi(x) = e^{\frac{1}{6}\pi i}\{\lambda - q(x)\}^{-\frac{1}{4}}\eta(x).$$

Accordingly, in $0 \leqslant x \leqslant X$, for sufficiently large (negative) λ,

$$\{\lambda-q(x)\}^{\frac14}\psi(x) = e^{\frac16\pi i}(\tfrac12\pi w)^{\frac12}H_{\frac13}^{(1)}(w)+O(\lambda^{-\frac32}e^{-\mathrm{im}\,w})+O(\lambda^{-\frac12}X^{-1}e^{-\mathrm{im}\,w}).$$

Further, for large $|w|$, with $\arg w = \frac32\pi$, (2.2 d) still holds, and, for $0 \leqslant x \leqslant \frac14 X$ and sufficiently large λ, $O(w^{-1})$ can be replaced by $O(\lambda^{-\frac12}X^{-1})$. For

$$\begin{aligned}|w| &= \left|\int\limits_X^x \{\lambda-q(t)\}^{\frac12}\,dt\right| \\ &> \left|\int\limits_{\frac14 X}^{\frac12 X} \{\lambda-q(t)\}^{\frac12}\,dt\right| \\ &> \tfrac14 X|\lambda-q(\tfrac12 X)|^{\frac12} \\ &> \tfrac14 AX|\lambda|^{\frac12}, \quad \text{by (2.2 b) or (2.2 c).}\end{aligned}$$

Hence finally, in $0 \leqslant x \leqslant \frac14 X$ and for sufficiently large λ,

$$\{\lambda-q(x)\}^{\frac14}\psi(x) = e^{i(w-\frac14\pi)}\{1+O(\lambda^{-\frac32})+O(\lambda^{-\frac12}X^{-1})\}. \tag{2.3}$$

We can similarly obtain solutions based on X_1 instead of on X. For $x \leqslant X$, we define $w_1(x)$ by (1.5), together with

$$\arg w_1 = \begin{cases}\frac32\pi & (x > X_1),\\ 0 & (x < X_1).\end{cases}$$

Then, by following through the same analysis as before, we find that, since, as $x \to -\infty$,

$$\{\lambda-q(x)\}^{\frac14}\psi(x) \sim \alpha\exp\{i(w_1-\tfrac14\pi)\}+\beta\exp\{-i(w_1-\tfrac14\pi)\},$$

we must have, uniformly for $-\infty < x \leqslant X_1$,

$$\{\lambda-q(x)\}^{\frac14}\psi(x) = \alpha e^{\frac16\pi i}(\tfrac12\pi w_1)^{\frac12}H_{\frac13}^{(1)}(w_1)+\beta e^{-\frac16\pi i}(\tfrac12\pi w_1)^{\frac12}H_{\frac13}^{(2)}(w_1)+$$
$$+O\left\{\int\limits_{-\infty}^{x} |g(t)|\,|\lambda-q(t)|^{\frac12}\,dt\right\},$$

where $g(t)$ is just $f(t)$ with w_1 replacing w. Now, for $X_1 < x \leqslant X$, we have $\arg w_1 = \frac32\pi$, and, for large $|w_1|$, with $\arg w_1 = \frac32\pi$, we have once again that (2.2 d) holds (with w_1 replacing w), and also that

$$(\tfrac12\pi w_1)^{\frac12}H_{\frac13}^{(2)}(w_1) = \{e^{-i(w_1-\frac16\pi-\frac14\pi)}+e^{i(w_1+\frac16\pi+\frac34\pi)}\}\{1+O(w_1^{-1})\}.$$

Hence, as before, we have finally that, in $\frac14 X_1 \leqslant x \leqslant 0$ and for sufficiently large λ,

$$\{\lambda-q(x)\}^{\frac14}\psi(x)$$
$$= \{(\alpha-\beta)e^{i(w_1-\frac14\pi)}+\beta e^{-i(w_1-\frac14\pi)}\}\{1+O(\lambda^{-\frac32})+O(\lambda^{-\frac12}X_1^{-1})\}. \tag{2.4}$$

The results obtained from (2.3) and (2.4) by formal differentiation also hold, and we have, in $0 \leqslant x \leqslant \frac{1}{4}X$,

$$\{\lambda - q(x)\}^{-\frac{1}{4}}\psi'(x) = ie^{i(w-\frac{1}{4}\pi)}\{1+O(\lambda^{-\frac{3}{2}})+O(\lambda^{-\frac{1}{2}}X^{-1})\},$$

while, in $\frac{1}{4}X_1 \leqslant x \leqslant 0$,

$$-\{\lambda - q(x)\}^{-\frac{1}{4}}\psi'(x)$$
$$= \{i(\alpha-\beta)e^{i(w_1-\frac{1}{4}\pi)} - i\beta e^{-i(w_1-\frac{1}{4}\pi)}\}\{1+O(\lambda^{-\frac{3}{2}})+O(\lambda^{-\frac{1}{2}}X_1^{-1})\}.$$

At $x = 0$, the Wronskian of the two different expressions obtained for $\psi(x)$ must vanish, i.e. at $x = 0$

$$e^{i(w-\frac{1}{4}\pi)}\{i(\alpha-\beta)e^{i(w_1-\frac{1}{4}\pi)} - i\beta e^{-i(w_1-\frac{1}{4}\pi)}\}\{1+O(\lambda^{-\frac{3}{2}})+O(\lambda^{-\frac{1}{2}}X^{-1})+O(\lambda^{-\frac{1}{2}}X_1^{-1})\}$$
$$= -ie^{i(w-\frac{1}{4}\pi)}\{(\alpha-\beta)e^{i(w_1-\frac{1}{4}\pi)} + \beta e^{-i(w_1-\frac{1}{4}\pi)}\}\{1+O(\lambda^{-\frac{3}{2}})+O(\lambda^{-\frac{1}{2}}X^{-1})+$$
$$+O(\lambda^{-\frac{1}{2}}X_1^{-1})\}.$$

Hence

$$2(\alpha-\beta)e^{i(w+w_1-\frac{1}{2}\pi)}\{1+O(\lambda^{-\frac{3}{2}})+O(\lambda^{-\frac{1}{2}}X^{-1})+O(\lambda^{-\frac{1}{2}}X_1^{-1})\}$$
$$= \beta e^{i(w-w_1)}\{O(\lambda^{-\frac{3}{2}})+O(\lambda^{-\frac{1}{2}}X^{-1})+O(\lambda^{-\frac{1}{2}}X_1^{-1})\},$$

so that
$$\alpha-\beta = O[\beta e^{-2iw_1}\{|\lambda^{-\frac{3}{2}}|+|\lambda^{-\frac{1}{2}}X^{-1}|+|\lambda^{-\frac{1}{2}}X_1^{-1}|\}]. \tag{2.5}$$

But $\psi(x)$ must also be continuous at $x = 0$, and so, substituting (2.5) in the condition for continuity, we obtain

$$\beta = e^{i(w+w_1-\frac{1}{2}\pi)}\{1+O(\lambda^{-\frac{3}{2}})+O(\lambda^{-\frac{1}{2}}X^{-1})+O(\lambda^{-\frac{1}{2}}X_1^{-1})\}. \tag{2.6}$$

Since
$$w+w_1 = \int_{X}^{X_1} \{\lambda - q(t)\}^{\frac{1}{2}}\, dt = -i\kappa,$$

(2.6) gives the required expression for $1/|\beta|^2$.

REFERENCES

1. E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations* (Oxford, 1946).
2. P. Morse and H. Feshbach, *Methods of theoretical physics*, vol. 2 (New York, 1953).
3. E. C. Titchmarsh, 'On the asymptotic distribution of eigenvalues', *Quart. J. of Math.* (Oxford) (2) 5 (1954) 228–40.
4. —— 'Some properties of eigenfunction expansions', ibid. 59–70.

A NOTE ON FREE MODULES OVER THE STEENROD ALGEBRA

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LET p be a prime, and let A denote the mod p Steenrod algebra [for the definition and properties of A cf. (**1**) and (**4**)]. By 'A-module' we shall always mean 'unitary left A-module'.

The importance of free A-modules has been emphasized by their occurrence in J. F. Adams' spectral sequence (**1**) and in his work on the non-existence of elements of Hopf invariant one. It is known that, if M is a free A-module, then any direct summand of M is also free; in fact, this follows from Lemma 1 below and a theorem of Kaplansky [(**3**) Theorem 2]. We shall prove a partial converse:

PROPOSITION. *If M is a free A-module, then any finitely-generated free submodule of M is a direct summand.*

We shall need only the following properties of the Steenrod algebra: A is an associative algebra over a field, with a unit 1, which is

(1) graded: i.e. as a vector space A is the direct sum of a sequence of subspaces A_0, A_1,... such that, if $a \in A_i$, $b \in A_j$, then $ab \in A_{i+j}$;

(2) connected: i.e. A_0 is generated as a vector space by 1.

(3) Any finite set of elements in A generates a finite-dimensional subalgebra.

LEMMA 1. *A is a local ring, in the sense that the non-units form a two-sided ideal $I(A)$.*

LEMMA 2. *Every finitely-generated proper right ideal in A has a non-zero left annihilator.*

The proposition follows from Lemma 2 and a theorem due to Bass [(**2**) Theorem 5.4].

Proof of Lemma 1. Set $I(A) = \sum_{n>0} A_n$. Clearly $I(A)$ is a two-sided ideal. It follows from (1) and (3) that every element of $I(A)$ is nilpotent and is thus a non-unit. It remains to prove that any other element of A is a unit. Such an element can be written as $\lambda+a$, where $\lambda \in A_0$, $a \in I(A)$, and $\lambda \neq 0$; and, by (2), it is sufficient to consider the case $\lambda = 1$. Since a is nilpotent, the series $1-a+a^2-a^3+...$ reduces to a finite sum, and yields an inverse for $1+a$.

Proof of Lemma 2. Let J be a proper right ideal in A, generated by a finite set of elements $b_1,\ldots, b_m$. Since J is proper, from Lemma 1, each b_i must lie in $I(A)$. Let $c_1,\ldots, c_r$ in $I(A)$ be the set of homogeneous components of the b's. There is a finite-dimensional graded subalgebra C of A which contains $c_1,\ldots, c_r$; according to (3), the graded subalgebra generated by the c's is such. Let d in C be a non-zero homogeneous element of maximal gradation. Then d annihilates every homogeneous element of positive gradation in C: in particular, $c_1,\ldots, c_r$. Thus d is a left annihilator for J.

This completes the proof of the proposition.

It is shown in (**4**) that the Steenrod algebra is actually the union of an ascending sequence of finite-dimensional graded subalgebras $\mathscr{S}^*(1) \subset \mathscr{S}^*(2) \subset \ldots$, so that the subalgebra C of Lemma 2 can be taken to be $\mathscr{S}^*(n)$ for some sufficiently large n, depending on $b_1,\ldots, b_m$. Then (**4**) Proposition 2 implies that the following is a non-zero homogeneous element of $\mathscr{S}^*(n)$ of maximal gradation: in Milnor's notation, if $p = 2$, $d = Sq^R$, while, if $p \neq 2$, $d = Q_0 \ldots Q_n \mathscr{P}^R$, where

$$R = (p^n-1, p^{n-1}-1,\ldots, p-1, 0, 0,\ldots).$$

I am indebted to the referee for clarifying the presentation of this note.

REFERENCES

1. J. F. Adams, 'On the structure and applications of the Steenrod algebra', *Comm. Math. Helvetici* 32 (1958) 180–214.
2. H. Bass, 'Finitistic dimension and a homological generalization of semi-primary rings', *Trans. American Math. Soc.* 95 (1960) 466–88.
3. I. Kaplansky, 'Projective modules', *Annals of Math.* 68 (1958) 372–7.
4. J. W. Milnor, 'The Steenrod algebra and its dual', ibid. 67 (1958) 150–71.